WEAK AND STRONG FORMS OF $\gamma$-IRRESOLUTENESS

ABSTRACT. In this paper we consider new weak and strong forms of $\gamma$-irresoluteness and $\gamma$-closure via the concept of $g\gamma$-closed sets which we call ap-$\gamma$-irresolute, ap-$\gamma$-closed and contra-$\gamma$-irresolute maps. Moreover, we use ap-$\gamma$-irresolute and ap-$\gamma$-closed maps to obtain a characterization of $\gamma-T_{1\frac{1}{2}}$-spaces.

KEY WORDS: topological spaces, generalized $\gamma$-closed sets, $\gamma$-open sets, $\gamma$-closed maps, $\gamma$-irresolute maps.

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1. Introduction and preliminaries

A.A. El-Atik [6] introduced the notion of $\gamma$-open sets and $\gamma$-continuity in topological spaces. Andrijevic [1] defined and investigated $b$-open sets which are equivalent with $\gamma$-open sets. El-Atik [6] introduced a new map called $\gamma$-irresolute which is contained in the class of $\gamma$-continuous maps. In this paper, we introduce weak and strong forms of $\gamma$-irresoluteness called ap-$\gamma$-irresoluteness and ap-$\gamma$-closedness by using $g\gamma$-closed sets and obtain some basic properties of such maps. This definition enables us to obtain conditions under which maps and inverse maps preserve $g\gamma$-closed sets. Also, in this paper we present a new generalization of contra $\gamma$-continuity due to the present Author and EL-Maghrabi [14, 7] called contra-$\gamma$-irresoluteness. We define this last class of maps by the requirement that the inverse of each $\gamma$-open set in the codomain is $\gamma$-closed in the domain. This notion is a stronger form of ap-$\gamma$-irresoluteness. Finally, we characterize the class of $\gamma-T_{1\frac{1}{2}}$ spaces in terms of ap-$\gamma$-irresolute and ap-$\gamma$-closed maps.

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of $(X, \tau)$. The subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-open [6] or $b$-open [1] or $sp$-open [5] (resp. $\alpha$-open [15], semi-open [10]) if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ (resp. $A \subseteq Int(Cl(Int(A)))$, $A \subseteq Cl(Int(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of $A$ respectively. The complement of a $\gamma$-open (resp. $\alpha$-open, semi-open) set is called $\gamma$-closed (resp. $\alpha$-closed, semi-closed). The intersection
of all $\gamma$-closed (resp. $\alpha$-closed, semi-closed) sets containing $A$ is called the $\gamma$-closure (res. $\alpha$-closure, semi-closure) of $A$ and is denoted by $\gamma Cl(A)$ resp. $\alpha Cl(A), sCl((A))$. The interior of $A$ is the union of all $\gamma$-open sets in $X$ and is denoted by $\gamma Int(A)$. The family of all $\gamma$-open (resp. $\gamma$-closed, $\alpha$-open, semi-open) sets in $X$ (resp. $\gamma C(X, \tau), \alpha O(X, \tau), SO(X, \tau)$) is denoted by $\gamma O(X, \tau)$. A subset $A$ of $(X, \tau)$ is said to be:

(i) generalized closed (briefly, $g$-closed) [11] set if $\gamma O(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$,

(ii) generalized $\alpha$-closed (briefly, $g\alpha$-closed) [12] set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$,

(iii) generalized semi-closed (briefly, $gs$-closed) [2] set if $\gamma cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$,

(iv) semi-generalized closed (briefly, $sg$-closed) [3] set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$,

(v) generalized $\gamma$-closed (briefly, $g\gamma$-closed) [8] (equivalently, $gb$-closed) [9] set if $\gamma cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\gamma$-open in $(X, \tau)$.

It should be noted that this notion is a particular case of the notion of generalized $(m_1, m_2)$-closed sets introduced by Noiri [16]. A subset $B$ is said to be generalized $\gamma$-open (briefly, $g\gamma$-open) in $(X, \tau)$ [8] if its complement $B^c = X - B$ is $g\gamma$-closed in $(X, \tau)$.

A map $f : (X, \tau) \to (Y, \sigma)$ is called:

(i) $\gamma$-irresolute [6] if for each $V \in \gamma O(Y, \sigma), f^{-1}(V) \in \gamma O(X, \tau)$.

(ii) pre-$\gamma$-closed [6] (resp. pre-$\gamma$-open [6]), if for every $\gamma$-closed (resp. $\gamma$-open) set $A$ of $(X, \tau)$, $f(A)$ is $\gamma$-closed (resp. $\gamma$-open) in $(Y, \sigma)$.

(iii) contra-$\gamma$-closed [7] if, $f(U)$ is $\gamma$-open in $Y$, for each closed set $U$ of $X$.

2. Ap-$\gamma$-irresolute, ap-$\gamma$-closed and contra-$\gamma$-irresolute maps

Definition 1. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately $\gamma$-irresolute (briefly, ap-$\gamma$-irresolute) if $\gamma Cl(A) \subseteq f^{-1}(G)$ whenever $G$ is a $\gamma$-open subset of $(Y, \sigma)$, $A$ is a $g\gamma$-closed subset of $(X, \tau)$ and $A \subseteq f^{-1}(G)$.

Definition 2. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately $\gamma$-closed (briefly, ap-$\gamma$-closed) if, $f(A) \subseteq \gamma Int(H)$ whenever $H$ is a $g\gamma$-open subset of $(Y, \sigma)$, $A$ is a $\gamma$-closed subset of $(X, \tau)$ and $f(A) \subseteq H$.

Theorem 1. (i) $f : (X, \tau) \to (Y, \sigma)$ is ap-$\gamma$-irresolute if $f^{-1}(G)$ is $\gamma$-closed in $(X, \tau)$, for every $G \in \gamma O(Y, \sigma)$.

(ii) $f : (X, \tau) \to (Y, \sigma)$ is ap-$\gamma$-closed if, $f(A) \in \gamma O(Y, \sigma)$, for every $\gamma$-closed subset $A$ of $(X, \tau)$. 
Proof. (i) Let $A \subseteq f^{-1}(G)$, where $G \in \gamma\mathcal{O}(Y, \sigma)$ and $A$ is a $g\gamma$-closed subset of $(X, \tau)$. Therefore $\gamma\text{Cl}(A) \subseteq \gamma\text{Cl}(f^{-1}(G)) = f^{-1}(G)$. Thus $f$ is ap-$\gamma$-irresolute.

(ii) Let $f(A) \subseteq H$, where $A$ is a $\gamma$-closed subset of $(X, \tau)$ and $H$ is a $g\gamma$-open subset of $(Y, \sigma)$. Therefore $\gamma\text{Int}(f(A)) \subseteq \gamma\text{Int}(H)$. Then $f(A) \subseteq \gamma\text{Int}(H)$. Thus $f$ is ap-$\gamma$-closed. ■

Clearly, $\gamma$-irresolute maps are ap-$\gamma$-irresolute. Also, pre-$\gamma$-closed maps are ap-$\gamma$-closed. The converse implications do not hold as it is shown in the following example.

Example 1. Let $X = \{a, b\}$ be the Sierpinski space with the topology $\tau = \{X, \phi, \{a\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = b$ and $f(b) = a$. Since the image of every $\gamma$-closed set is $\gamma$-open, then $f$ is ap-$\gamma$-closed (similarly, since the inverse image of every $\gamma$-open set is $\gamma$-closed, then $f$ ap-$\gamma$-irresolute). However $\{b\}$ is $\gamma$-closed in $(X, \tau)$ (resp. $\{a\}$ is $\gamma$-open), but $f(\{b\})$ is not $\gamma$-closed (resp. $f^{-1}(\{a\})$ is not $\gamma$-open) in $(X, \tau)$. Therefore $f$ is not pre-$\gamma$-closed (resp. $f$ is not $\gamma$-irresolute).

Remark 1. Let be $(X, \tau)$ a space as defined in Example 1. Then the identity map on $(X, \tau)$ is both ap-$\gamma$-irresolute and ap-$\gamma$-closed. It is clear that the converses of (i) and (ii) in Theorem 1 do not hold.

In the following result, the converses of (i) and (ii) in Theorem 1 are true under certain conditions.

Theorem 2. Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a space $(X, \tau)$ to a space $(Y, \sigma)$.

(i) Let all subsets of $(X, \tau)$ be clopen, then $f$ is ap-$\gamma$-irresolute if and only if $f^{-1}(G)$ is $\gamma$-closed in $(X, \tau)$, for every $G \in \gamma\mathcal{O}(Y, \sigma)$,

(ii) Let all subsets of $(Y, \sigma)$ be clopen, then $f$ is ap-$\gamma$-closed if and only if $f(A) \in \gamma\mathcal{O}(Y, \sigma)$, for every $\gamma$-closed subset $A$ of $(X, \tau)$.

Proof. (i) The sufficiency is stated in Theorem 1.

Necessity. Assume that $f$ is ap-$\gamma$-irresolute. Let $A$ be an arbitrary subset of $(X, \tau)$ such that $A \subseteq H$, where $H \in \gamma\mathcal{O}(X, \tau)$. Then by hypothesis $\gamma\text{Cl}(A) \subseteq \gamma\text{Cl}(H) = H$. Therefore all subsets of $(X, \tau)$ are $g\gamma$-closed (hence and all are $g\gamma$-open). So, for any $G \in \gamma\mathcal{O}(Y, \sigma)$, $f^{-1}(G)$ is $\gamma$-closed in $(X, \tau)$. Since $f$ is ap-$\gamma$-irresolute, $\gamma\text{Cl}(f^{-1}(G)) \subseteq f^{-1}(G)$. Therefore $\gamma\text{Cl}(f^{-1}(G)) = f^{-1}(G)$, i.e., $f^{-1}(G)$ is $\gamma$-closed in $(X, \tau)$.

(ii) The sufficiency is clear by Theorem 1.

Necessity. Assume that $f$ is ap-$\gamma$-closed. As in (i), we obtain that all subsets of $(Y, \sigma)$ are $g\gamma$-open. Therefore for any $\gamma$-closed subset $A$ of $(X, \tau)$, $f(A)$ is $g\gamma$-open in $Y$. Since $f$ is ap-$\gamma$-closed $f(A) \subseteq \gamma\text{Int}(f(A))$. Hence $f(A) = \gamma\text{Int}(f(A))$, i.e., $f(A)$ is $\gamma$-open. ■
As an immediate consequence of Theorem 2, we have the following.

**Corollary 1.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a map from a topological space \((X, \tau)\) to a topological space \((Y, \sigma)\).

(i) Let all subsets of \((X, \tau)\) be clopen, then \( f \) is ap-\(\gamma\)-irresolute if and only if, \( f \) is \(\gamma\)-irresolute,

(ii) Let all subsets of \((Y, \sigma)\) be clopen, then \( f \) is ap-\(\gamma\)-closed if and only if, \( f \) is pre \(\gamma\)-closed.

**Definition 3.** A map \( f : (X, \tau) \to (Y, \sigma) \) is called:

(i) contra-\(\gamma\)-irresolute if \( f^{-1}(G) \) is \(\gamma\)-closed in \((X, \tau)\) for each \( G \in \gamma O(Y, \sigma) \),

(ii) contra-pre-\(\gamma\)-closed if \( f(A) \in \gamma O(Y, \sigma) \), for each \(\gamma\)-closed set \( A \) of \((X, \tau)\).

**Remark 2.** In fact, contra-\(\gamma\)-irresoluteness and \(\gamma\)-irresoluteness are independent notions. Example 1 shows that contra-\(\gamma\)-irresoluteness does not imply \(\gamma\)-irresoluteness while the converse is shown in the following example.

**Example 2.** A \(\gamma\)-irresolute map need not be contra-\(\gamma\)-irresolute. The identity map on the topological space \((X, \tau)\), where \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( X = \{a, b, c\} \) is an example of a \(\gamma\)-irresolute map which is not contra-\(\gamma\)-irresolute.

Recall that a map \( f : (X, \tau) \to (Y, \sigma) \) is contra-\(\gamma\)-continuous [7, 14] if, \( f^{-1}(G) \) is \(\gamma\)-closed in \((X, \tau)\), for each open set \( G \) of \((Y, \sigma)\).

Every contra-\(\gamma\)-irresolute map is contra-\(\gamma\)-continuous, but not conversely as the following example shows.

**Example 3.** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( Y = \{p, q\} \), \( \sigma = \{Y, \phi, \{p\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = p \) and \( f(b) = f(c) = q \). Then \( f \) is contra-\(\gamma\)-continuous, but \( f \) is not contra-\(\gamma\)-irresolute.

The following result can be easily verified. Therefore we omitted its proof.

**Theorem 3.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a map. Then the following conditions are equivalent:

(i) \( f \) is contra-\(\gamma\)-irresolute,

(ii) The inverse image of each \(\gamma\)-closed set of \( Y \) is \(\gamma\)-open in \( X \).

**Remark 3.** By Theorem 1, we have that every contra-\(\gamma\)-irresolute map is ap-\(\gamma\)-irresolute and every contra-\(\gamma\)-closed map is ap-\(\gamma\)-closed, the converse implications do not hold (see Remark 1).

A map \( f : (X, \tau) \to (Y, \sigma) \) is called perfectly contra-\(\gamma\)-irresolute if the inverse of every \(\gamma\)-open set of \( Y \) is \(\gamma\)-clopen in \( X \).

**Lemma 1.** Every perfectly contra-\(\gamma\)-irresolute map is contra-\(\gamma\)-irresolute and \(\gamma\)-irresolute. But the converse may not be true.
Example 4. Remark 2 is an example of a contra-γ-irresolute map which is not perfectly contra-γ-irresolute and Example 3 is an example of a γ-irresolute map which is not perfectly contra-γ-irresolute.

Remark 4. For the definitions of ap-irresolute (resp. ap-α-irresolute), contra-irresolute (resp. contra-α-irresolute), perfectly contra-irresolute (resp. perfectly contra-α-irresolute) and irresolute (resp. α-irresolute) see [3, 4, 13].

Example 5. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_3 = \{\phi, X\}$. Then,

$SO(X, \tau) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\} = \gamma O(X, \tau),$

$\alpha O(X, \tau) = \{\phi, X, \{a\}, \{a, b\}\},$

$SO(X, \tau_1) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\} = \alpha O(X, \tau_1) = \gamma O(X, \tau_1),$

$SO(X, \tau_2) = \{\phi, X, \{c\}, \{a, b\}\} = \alpha O(X, \tau_2),$

$\gamma O(X, \tau_2) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}. Then,

(a) Let $f: (X, \tau) \to (X, \tau_2)$ be defined as $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then:

(i) $f$ is contra-irresolute (hence, ap-irresolute), but $f$ is not contra-α-irresolute (hence $f$ is not perfectly contra-α-irresolute);

(ii) $f$ is irresolute but $f$ is not γ-irresolute;

(iii) $f$ is irresolute but $f$ is not α-irresolute.

(b) Let $f: (X, \tau_3) \to (X, \tau)$ be the identity map. Then:

(i) $f$ is γ-irresolute but $f$ is not irresolute;

(ii) $f$ is γ-irresolute but $f$ is not α-irresolute.

(c) Let $f: (X, \tau_2) \to (X, \tau_1)$ be the identity map. Then:

(i) $f$ is contra-γ-irresolute but $f$ is not contra-irresolute;

(ii) $f$ is contra-γ-irresolute but $f$ is not contra-α-irresolute;

(d) Let $f: (X, \tau_2) \to (X, \tau_2)$ be the identity map. Then:

(i) $f$ is perfectly contra-γ-irresolute but $f$ is not perfectly contra-irresolute;

(ii) $f$ is perfectly contra-γ-irresolute but $f$ is not perfectly contra-α-irresolute.

Example 6. EL-Atik [6] For any countable set $X$, the identity maps from an indiscrete space into any other one is γ-irresolute but it is not irresolute.

Example 7. EL-Atik [6] The identity function from a particular point topological space on any countable set with any particular point into an indiscrete one is irresolute but not γ-irresolute.

Clearly, the following diagram holds and none of its implications are reversible:
The following theorem is a decomposition of perfectly contra-γ-irresoluteness.

**Theorem 4.** For a function \(f : (X, \tau) \to (Y, \sigma)\), the following conditions are equivalent:

(i) \(f\) is perfectly contra-γ-irresolute,
(ii) \(f\) is contra-γ-irresolute and γ-irresolute.

**Theorem 5.** If a map \(f : (X, \tau) \to (Y, \sigma)\) is γ-irresolute and ap-γ-closed, then \(f^{-1}(A)\) is \(g\gamma\)-closed (resp. \(g\gamma\)-open) whenever \(A\) is a \(g\gamma\)-closed (resp. \(g\gamma\)-open) subset of \((Y, \sigma)\).

**Proof.** Let \(A\) be a \(g\gamma\)-closed subset of \((Y, \sigma)\). Suppose that \(f^{-1}(A) \subseteq G\) where \(G \in \gamma O(X, \tau)\). Taking complements, we obtain \(G^c \subseteq f^{-1}(A^c)\) or \(f(G^c) \subseteq A^c\). Since \(f\) is ap-γ-closed, then \(f(G^c) \subseteq \gamma Int(A^c) = (\gamma Cl(A))^c\). It follows that \(G^c \subseteq (f^{-1}(\gamma Cl(A)))^c\) and hence \(f^{-1}(\gamma Cl(A)) \subseteq G\). Since \(f\) is γ-irresolute, \(f^{-1}(\gamma Cl(A))\) is γ-closed. Thus we have

\[
\gamma Cl(f^{-1}(A)) \subseteq \gamma Cl(f^{-1}(\gamma Cl(A))) = f^{-1}(\gamma Cl(A)) \subseteq G.
\]

This implies that \(f^{-1}(A)\) is \(g\gamma\)-closed in \((X, \tau)\). 

A similar argument shows that inverse images of \(g\gamma\)-open sets are \(g\gamma\)-open.
**Theorem 6.** If a map \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha\gamma\)-irresolute and \( \alpha\gamma \)-closed, then for every \( g\gamma \)-closed subset \( V \) of \((X, \tau)\) \( f(V) \) is a \( g\gamma \)-closed set of \((Y, \sigma)\).

**Proof.** Let \( V \) be a \( g\gamma \)-closed subset of \((X, \tau)\). Let \( f(V) \subseteq G \) where \( G \in \gamma O(Y, \sigma) \). Then \( V \subseteq f^{-1}(G) \) holds. Since \( f \) is \( \alpha\gamma \)-irresolute, \( \gamma Cl(V) \subseteq (f^{-1}(G)) \) and hence \( f(\gamma Cl(V)) \subseteq G \). Therefore, we have \( \gamma Cl(f(V)) \subseteq \gamma Cl((\gamma Cl(V)) = f(\gamma Cl(V)) \subseteq G \). Hence \( f(V) \) is \( g\gamma \)-closed in \((Y, \sigma)\).

It should be noticed that the composition of two \( \alpha\gamma \)-irresolute maps need not be \( \alpha\gamma \)-irresolute. Let \( X = \{a, b\} \) be the Sierpinski space and set \( \tau = \{\phi, X, \{a\}\} \) and \( \sigma = \{\phi, X, \{b\}\} \). The identity maps \( f : (X, \tau) \to (X, \sigma) \) and \( g : (X, \sigma) \to (X, \tau) \) are both \( \alpha\gamma \)-irresolute but their composition \( g \circ f : (X, \tau) \to (X, \tau) \) is not \( \alpha\gamma \)-irresolute.

However the following theorem holds, the proof is easy and hence omitted.

**Theorem 7.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \to (Z, \eta) \). Then:

(i) \( g \circ f \) is \( \alpha\gamma \)-irresolute, if \( g \) is \( \gamma \)-irresolute and \( f \) is \( \alpha\gamma \)-irresolute;

(ii) \( g \circ f \) is \( \alpha\gamma \)-irresolute, if \( g \) is \( \alpha\gamma \)-irresolute and \( f \) is \( \gamma \)-irresolute.

In analogous way, we have the following.

**Theorem 8.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \to (Z, \eta) \). Then:

(i) \( g \circ f \) is \( \alpha\gamma \)-closed, if \( g \) is \( \alpha\gamma \)-closed and \( f \) is \( \alpha\gamma \)-closed;

(ii) \( g \circ f \) is \( \alpha\gamma \)-closed, if \( g \) is \( \gamma \)-open, \( f \) is \( \alpha\gamma \)-closed and \( g^{-1} \) preserves \( \gamma \)-open sets;

(iii) \( g \circ f \) is \( \alpha\gamma \)-irresolute, if \( g \) is \( \gamma \)-irresolute and \( f \) is \( \alpha\gamma \)-irresolute.

**Proof.** (i) Suppose that \( A \) is an arbitrary \( \gamma \)-closed subset of \((X, \tau)\) and \( B \) is a \( \gamma \)-open subset of \((Z, \eta)\) for which \((g \circ f)(A) \subseteq B \). Then \( f(A) \) is \( \gamma \)-closed in \((Y, \sigma)\), because \( f \) is \( \alpha\gamma \)-closed. Since \( g \) is \( \alpha\gamma \)-closed, \( g(f(A)) \subseteq \gamma - Int(B) \). This implies that \( g \circ f \) is \( \alpha\gamma \)-closed.

(ii) Suppose that \( A \) is an arbitrary \( \gamma \)-closed subset of \((X, \tau)\) and \( B \) is a \( \gamma \)-open subset of \((Z, \eta)\) for which \((g \circ f)(A) \subseteq B \). Hence \( f(A) \subseteq g^{-1}(B) \). Because, \( g^{-1}(B) \) is \( \gamma \)-open and \( f \) is \( \alpha\gamma \)-closed. Thus \((g \circ f)(A) = g(f(A)) \subseteq g(\gamma - Int(g^{-1}(B)) \subseteq \gamma Int(gg^{-1}(B)) \subseteq \gamma Int(B) \). This implies that \( g \circ f \) is \( \alpha\gamma \)-closed.

(iii) Suppose that \( A \) is an arbitrary \( \gamma \)-closed subset of \((X, \tau)\) and \( G \in \gamma O(Z, \eta) \) for which \( A \subseteq (g \circ f)^{-1}(G) \). Then \( g^{-1}(G) \in \gamma O(Y, \sigma) \) because \( g \) is \( \gamma \)-irresolute. Since \( f \) is \( \alpha\gamma \)-irresolute, \( \gamma Cl(A) \subseteq f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \). This proves that \( g \circ f \) is \( \alpha\gamma \)-irresolute.

\[ \blacksquare \]
As a consequence of Theorem 8, we have.

**Corollary 2.** Let \( f_\alpha : X \to Y_\alpha \) be a map for each \( \alpha \in \Omega \) and let \( f : X \to \Pi Y_\alpha \) be the product map given by \( f(x) = (f_\alpha(x)) \). If, \( f \) is ap-\( \gamma \)-irresolute, then \( f_\alpha \) is ap-\( \gamma \)-irresolute for each \( \alpha \).

**Proof.** For each \( \gamma \), let \( P_\gamma : \Pi Y_\alpha \to Y_\gamma \) be the projection map. Then \( f_\gamma = P_\gamma \circ f \), where \( P_\gamma \) is \( \gamma \)-irresolute. By Theorem 8(iii) \( f_\gamma \) is ap-\( \gamma \)-irresolute. \( \blacksquare \)

**Lemma 2.** Let \( A \) and \( Y \) be subsets of a space \( X \). If \( A \in \gamma O(Y, \tau_Y) \) and \( Y \in \gamma O(X, \tau) \), then \( A \in \gamma O(X, \tau) \).

**Lemma 3.** Let \( X \) be a topological space and \( A, Y \) be subsets of \( X \) such that \( A \subseteq Y \subseteq X \) and \( Y \in \gamma O(X, \tau) \). Then \( \gamma Cl(A) \cap Y = \gamma Cl_Y(A) \), where \( \gamma Cl_Y(A) \) denotes the \( \gamma \)-closure of \( A \) in the subspace \( Y \).

Regarding the restriction \( f_A \) of a map \( f : (X, \tau) \to (Y, \sigma) \) to a subset \( A \) of \( X \), we have the following.

**Theorem 9.** (i) If \( f : (X, \tau) \to (Y, \sigma) \) is ap-\( \gamma \)-closed and \( A \) is a \( \gamma \)-closed set of \( (X, \tau) \), then its restriction \( f_A : (A, \tau_A) \to (Y, \sigma) \) is ap-\( \gamma \)-closed;

(ii) If, \( f : (X, \tau) \to (Y, \sigma) \) is ap-\( \gamma \)-irresolute and \( A \) is an open, \( g\gamma \)-closed subset of \( (X, \tau) \), then its restriction \( f_A : (A, \tau_A) \to (Y, \sigma) \) is ap-\( \gamma \)-irresolute.

**Proof.** (i) Suppose that \( B \) is arbitrary \( \gamma \)-closed subset of \( (A, \tau_A) \) and \( G \) is a \( g\gamma \)-open subset of \( (Y, \sigma) \) for which \( f_A(B) \subseteq G \). By Lemma 2, \( B \) is \( \gamma \)-closed subset of \( (X, \tau) \). Since \( A \) is a \( \gamma \)-closed subset of \( (X, \tau) \), then \( f_A(B) = f(B) \subseteq G \). Using Definition 2, we have \( f_A(B) \subseteq \gamma Int(G) \). Thus \( f_A \) is an ap-\( \gamma \)-closed map.

(ii) Assume that \( V \) is a \( g\gamma \)-closed subset relative to \( A \), i.e., \( V \) is \( g\gamma \)-closed in \( (A, \tau_A) \) and \( G \) is a \( \gamma \)-open subset of \( (Y, \sigma) \) for which \( V \subseteq (f_A)^{-1}(G) \). Then \( V \subseteq f^{-1}(G) \cap A \).

On the other hand, \( V \) is \( g\gamma \)-closed in \( X \). Since \( f \) is ap-\( \gamma \)-irresolute, then \( \gamma Cl(V) \subseteq f^{-1}(G) \). This implies that \( \gamma Cl(V) \cap A \subseteq f^{-1}(G) \cap A \). Using the fact that \( \gamma Cl(V) \cap A = \gamma Cl_A(V) \) (Lemma 3), we have \( \gamma Cl_A(V) \subseteq (f_A)^{-1}(G) \). Thus \( f_A : (A, \tau_A) \to (Y, \sigma) \) is ap-\( \gamma \)-irresolute. \( \blacksquare \)

### 3. Characterizations of \( \gamma-T_{\frac{1}{2}} \)-spaces

In the following result, we offer a characterization of the class of \( \gamma-T_{\frac{1}{2}} \)-spaces by using the concepts of ap-\( \gamma \)-irresolute and ap-\( \gamma \)-closed maps.

**Definition 4.** A space \((X, \tau)\) is said to be \( \gamma-T_{\frac{1}{2}} \)-space, if every \( g\gamma \)-closed set is \( \gamma \)-closed.
Theorem 10. Let \((X, \tau)\) be a space. Then the following statements are equivalent.

(i) \((X, \tau)\) is a \(\gamma - T_{\frac{1}{2}}\)-space;

(ii) \(f\) is ap-\(\gamma\)-irresolute, for every space \((Y, \sigma)\) and every map \(f : (X, \tau) \to (Y, \sigma)\).

Proof. (i) \(\to\) (ii). Let \(V\) be a \(g\gamma\)-closed subset of \((X, \tau)\) and \(V \subseteq f^{-1}(G)\), where \(G \in \gamma O(Y, \sigma)\). Since \((X, \tau)\) is a \(\gamma - T_{\frac{1}{2}}\)-space, \(V\) is \(\gamma\)-closed (i.e., \(V = \gamma Cl(V)\)). Therefore \(\gamma Cl(V) \subseteq f^{-1}(G)\) and hence \(f\) is ap-\(\gamma\)-irresolute.

(ii) \(\to\) (i). Let \(B\) be a \(g\gamma\)-closed subset of \((X, \tau)\) and \(Y\) be the set \(X\) with the topology \(\sigma = \{\phi, Y, B\}\). Finally let \(f : (X, \tau) \to (Y, \sigma)\) be the identity map. By the assumption \(f\) is ap-\(\gamma\)-irresolute. Since \(B\) is \(g\gamma\)-closed in \((X, \tau)\) and \(\gamma\)-open in \((Y, \sigma)\) and \(B \subseteq f^{-1}(B)\), it follows that \(\gamma Cl(B) \subseteq f^{-1}(B) = B\). Hence \(B\) is \(\gamma\)-closed in \((X, \tau)\). Therefore \((X, \tau)\) is a \(\gamma - T_{\frac{1}{2}}\)-space. ■

Theorem 11. Let \((Y, \sigma)\) be a space. Then the following statements are equivalent.

(i) \((Y, \sigma)\) is a \(\gamma - T_{\frac{1}{2}}\)-space;

(ii) \(f\) is ap-\(\gamma\)-closed, for every space \((X, \tau)\) and every map \(f : (X, \tau) \to (Y, \sigma)\).

Proof. This is analogous to the proof of Theorem 10. ■

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