ON ITERATE MINIMAL STRUCTURES AND M-ITERATE CONTINUOUS FUNCTIONS

Abstract. We introduce the notion of \(m\)IT-structures determined by operators \(m\)Int and \(m\)Cl on an \(m\)-space \((X, m_X)\). By using \(m\)IT-structures, we introduce and investigate a function \(f : (X, m\)IT) \(\rightarrow (Y, m_Y)\) called MIT-continuous. As special cases of MIT-continuity, we obtain \(M\)-semicontinuity [21] and \(M\)-precontinuity [23].

Key words: \(m\)-structure, \(M\)-continuous, \(m\)-semiopen, \(m\)-preopen, \(m\)IT-structure, MIT-continuous.

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1. Introduction

Semi-open sets, preopen sets, \(\alpha\)-open sets, \(\beta\)-open sets and \(b\)-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets, several authors introduced and studied various types of non-continuous functions. Certain of these non-continuous functions have properties similar to those of continuous functions and they hold, in many part, parallel to the theory of continuous functions.

In [26] and [27], the present authors introduced and studied the notions of minimal structures, \(m\)-spaces, \(m\)-continuity and \(M\)-continuity. Quite recently, in [19], [20] and [22], Min and Kim introduced the notions of \(m\)-semiopen sets, \(m\)-preopen sets and \(\alpha m\)-open sets which generalize the notion of \(m\)-open sets and also \(M\)-semicontinuity, \(M\)-precontinuity and \(\alpha M\)-continuity which generalize the notion of \(M\)-continuity. Rosas et al. [30] also introduced the notions of \(m\)-semiopen sets and \(m\)-preopen sets. The notion of \(\beta m\)-open sets is introduced by Boonpok [5].

The notions of \(m\)-semiopen sets, \(m\)-preopen sets, \(\alpha m\)-open sets and \(\beta m\)-open sets are defined by using the \(m\)-interior \(m\)Int and the \(m\)-closure \(m\)Cl on an \(m\)-space \((X, m_X)\). The each family of \(m\)-semiopen sets, \(m\)-preopen sets, \(\alpha m\)-open sets or \(\beta m\)-open sets becomes an \(m\)-structure with property \(B\), that is, it is closed under arbitrary union. The purpose of the present
paper is to obtain the unified theory of $M$-semicontinuity, $M$-precontinuity, $\alpha M$-continuity, $\beta M$-continuity and $M$-$b$-continuity.

2. Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

**Definition 1.** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be

(a) $\alpha$-open [24] if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$,
(b) semi-open [11] if $A \subset \text{Cl}(\text{Int}(A))$,
(c) preopen [16] if $A \subset \text{Int}(\text{Cl}(A))$,
(d) $b$-open [4] or $\gamma$-open [9] if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$,
(e) $\beta$-open [1] or semi-preopen [3] if $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$.

The family of all $\alpha$-open (resp. semi-open, preopen, $b$-open, $\beta$-open) sets in $(X, \tau)$ is denoted by $\alpha(X)$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$, $\beta(X)$).

**Definition 2.** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be $\alpha$-closed [18] (resp. semi-closed [6], preclosed [16], $b$-closed [4], $\beta$-closed [1]) if the complement of $A$ is $\alpha$-open (resp. semi-open, preopen, $b$-open, $\beta$-open).

**Definition 3.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The intersection of all $\alpha$-closed (resp. semi-closed, preclosed, $b$-closed, $\beta$-closed) sets of $X$ containing $A$ is called the $\alpha$-closure [18] (resp. semi-closure [6], preclosure [10], $b$-closure [4], $\beta$-closure [2]) of $A$ and is denoted by $\alpha\text{Cl}(A)$ (resp. $s\text{Cl}(A)$, $p\text{Cl}(A)$, $b\text{Cl}(A)$, $\beta\text{Cl}(A)$).

**Definition 4.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The union of all $\alpha$-open (resp. semi-open, preopen, $b$-open, $\beta$-open) sets of $X$ contained in $A$ is called the $\alpha$-interior [18] (resp. semi-interior [6], preinterior [10], $b$-interior [4], $\beta$-interior [2]) of $A$ and is denoted by $\alpha\text{Int}(A)$ (resp. $s\text{Int}(A)$, $p\text{Int}(A)$, $b\text{Int}(A)$, $\beta\text{Int}(A)$).

**Definition 5.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be irresolute [7] (resp. preirresolute [29] or $M$-preirresolute [17], $\alpha$-irresolute [13] or strongly feebly continuous [12], $\gamma$-irresolute (= $b$-irresolute) [8], $\beta$-irresolute [14]) at $x \in X$ if for each semi-open (resp. preopen, $\alpha$-open, $\gamma$-open, $\beta$-open) set $V$ containing $f(x)$, there exists a semi-open (resp. preopen, $\alpha$-open, $\gamma$-open, $\beta$-open) set $U$ of $X$ containing $x$ such that $f(U) \subset V$. The function $f$ is said to be irresolute (resp. preirresolute, $\alpha$-irresolute, $\gamma$-irresolute, $\beta$-irresolute) if it has this property at each point $x \in X$. 
3. Minimal structures and $M$-continuity

**Definition 6.** Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ [26], [27] if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, m_X)$, we denote a nonempty set $X$ with an $m$-structure $m_X$ on $X$ and call it an $m$-space. Each member of $m_X$ is said to be $m_X$-open (briefly $m$-open) and the complement of an $m_X$-open set is said to be $m_X$-closed (briefly $m$-closed).

**Remark 1.** Let $(X, \tau)$ be a topological space. The families $\tau$, $\alpha(X)$, $\text{SO}(X)$, $\text{PO}(X)$, $\text{BO}(X)$ and $\beta(X)$ are all minimal structures on $X$.

**Definition 7.** Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_X$-closure of $A$ and the $m_X$-interior of $A$ are defined in [15] as follows:

(a) $m\text{Cl}(A) = \cap \{F : A \subset F, X \setminus F \in m_X \}$,
(b) $m\text{Int}(A) = \cup \{U : U \subset A, U \in m_X \}$.

**Remark 2.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\alpha(X)$, $\text{BO}(X)$, $\beta(X)$), then we have

(a) $m\text{Cl}(A) = \text{Cl}(A)$ (resp. $s\text{Cl}(A), p\text{Cl}(A), \alpha\text{Cl}(A), b\text{Cl}(A), \beta\text{Cl}(A)$),
(b) $m\text{Int}(A) = \text{Int}(A)$ (resp. $s\text{Int}(A), p\text{Int}(A), \alpha\text{Int}(A), b\text{Int}(A), \beta\text{Int}(A)$).

**Lemma 1** (Maki et al. [15]). Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:

(a) $m\text{Cl}(X \setminus A) = X \setminus m\text{Int}(A)$ and $m\text{Int}(X \setminus A) = X \setminus m\text{Cl}(A)$,
(b) If $(X \setminus A) \in m_X$, then $m\text{Cl}(A) = A$ and if $A \in m_X$, then $m\text{Int}(A) = A$,
(c) $m\text{Cl}(\emptyset) = \emptyset$, $m\text{Cl}(X) = X$, $m\text{Int}(\emptyset) = \emptyset$ and $m\text{Int}(X) = X$,
(d) If $A \subset B$, then $m\text{Cl}(A) \subset m\text{Cl}(B)$ and $m\text{Int}(A) \subset m\text{Int}(B)$,
(e) $A \subset m\text{Cl}(A)$ and $m\text{Int}(A) \subset A$,
(f) $m\text{Cl}(m\text{Cl}(A)) = m\text{Cl}(A)$ and $m\text{Int}(m\text{Int}(A)) = m\text{Int}(A)$.

**Lemma 2** (Popa and Noiri [26]). Let $(X, m_X)$ be an $m$-space and $A$ a subset of $X$. Then $x \in m\text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing $x$.

**Definition 8.** A minimal structure $m_X$ on a nonempty set $X$ is said to have property $\mathcal{B}$ [15] if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

**Remark 3.** If $(X, \tau)$ is a topological space, then the $m$-structures $\text{SO}(X)$, $\text{PO}(X)$, $\alpha(X)$, $\text{BO}(X)$ and $\beta(X)$ have property $\mathcal{B}$. 
Lemma 3 (Popa and Noiri [28]). Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:

(a) $A \in m_X$ if and only if $\text{mInt}(A) = A$,
(b) $A$ is $m_X$-closed if and only if $\text{mCl}(A) = A$,
(c) $\text{mInt}(A) \in m_X$ and $\text{mCl}(A)$ is $m_X$-closed.

Definition 9. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M$-continuous at $x \in X$ [26] if for each $m_Y$-open set $V$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$. The function $f$ is $M$-continuous if it has this property at each $x \in X$.

Theorem 1 (Popa and Noiri [26]). For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:

(a) $f$ is $M$-continuous;
(b) $f^{-1}(V) = \text{mInt}(f^{-1}(V))$ for every $m$-open set $V$ of $Y$;
(c) $f^{-1}(F) = \text{mCl}(f^{-1}(F))$ for every $m$-closed set $F$ of $Y$;
(d) $\text{mCl}(f^{-1}(B)) \subset f^{-1}(\text{mCl}(B))$ for every subset $B$ of $Y$;
(e) $f(\text{mCl}(A)) \subset \text{mCl}(f(A))$ for every subset $A$ of $X$;
(f) $f^{-1}(\text{mCl}(B)) \subset \text{mInt}(f^{-1}(B))$ for every subset $B$ of $Y$.

Corollary 1 (Popa and Noiri [26]). For a function $f : (X, m_X) \to (Y, m_Y)$, where $m_X$ has property $\mathcal{B}$, the following properties are equivalent:

(a) $f$ is $M$-continuous;
(b) $f^{-1}(V)$ is $m$-open in $X$ for every $m$-open set $V$ of $Y$;
(c) $f^{-1}(F)$ is $m$-closed in $X$ for every $m$-closed set $F$ of $Y$.

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_M(f)$ as follows:

$$D_M(f) = \{ x \in X : f \text{ is not } M \text{-continuous at } x \}.$$

Theorem 2 (Noiri and Popa [25]). For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties hold:

$$D_M(f) = \bigcup_{G \in m_Y} \{ f^{-1}(G) \cap \text{mInt}(f^{-1}(G)) \} = \bigcup_{B \in \mathcal{P}(Y)} \{ f^{-1}(\text{mInt}(B)) \cap \text{mInt}(f^{-1}(B)) \} = \bigcup_{B \in \mathcal{P}(Y)} \{ \text{mCl}(f^{-1}(B)) - f^{-1}(\text{mCl}(B)) \} = \bigcup_{A \in \mathcal{P}(X)} \{ \text{mCl}(A) - f^{-1}(\text{mCl}(f(A))) \} = \bigcup_{F \in \mathcal{F}} \{ \text{mCl}(f^{-1}(F)) - f^{-1}(F) \},$$

where $\mathcal{F}$ is the family of $m$-closed sets of $(Y, m_Y)$.
4. \( m \)-Iterate structures and \( M \)-iterate continuity

Definition 10. Let \((X, m_X)\) be an \( m \)-space. A subset \( A \) of \( X \) is said to be

(a) \( \alpha m \)-open [20] if \( A \subset m \text{Int}(m \text{Cl}(m \text{Int}(A))) \),

(b) \( m \)-semiopen [19] if \( A \subset m \text{Cl}(m \text{Int}(A)) \),

(c) \( m \)-preopen [22] if \( A \subset m \text{Int}(m \text{Cl}(A)) \),

(d) \( \beta m \)-open [5] if \( A \subset m \text{Cl}(m \text{Int}(m \text{Cl}(A))) \),

(e) \( m \)-\( b \)-open if \( A \subset m \text{Int}(m \text{Cl}(A)) \cup m \text{Cl}(m \text{Int}(A)). \)

The family of all \( \alpha m \)-open (resp. \( m \)-semiopen, \( m \)-preopen, \( \beta m \)-open, \( m \)-\( b \)-open) sets in \((X, m_X)\) is denoted by \( \alpha m(X) \) (resp. \( m \text{SO}(X) \), \( m \text{PO}(X) \), \( \beta m(X), m \text{BO}(X) \).)

Remark 4. Let \((X, m_X)\) be an \( m \)-space.

(a) Similar definitions of \( m \)-semiopen sets, \( m \)-preopen sets, \( \alpha m \)-open sets, \( \beta m \)-open sets are provided in [30].

(b) The families \( \alpha m(X), m \text{SO}(X), m \text{PO}(X), \beta m(X) \) and \( m \text{BO}(X) \) are all minimal structures on \( X \).

Let \((X, m_X)\) be an \( m \)-space. Then \( m \text{SO}(X), m \text{PO}(X), \alpha m(X), \beta m(X) \) and \( m \text{BO}(X) \) are determined by iterating operators \( m \text{Int} \) and \( m \text{Cl} \). Hence, they are called \( m \)-iterate structures and are denoted by \( m \text{IT}(X) \) (briefly \( m \text{IT} \)).

Remark 5. (a) It easily follows from Lemma 3.1(3)(4) that \( m \text{SO}(X), m \text{PO}(X), \alpha m(X), \beta m(X) \) and \( m \text{BO}(X) \) are minimal structures with property \( B \). They are also shown in Theorem 3.5 of [19], Theorem 3.4 of [22] and Theorem 3.4 of [20] for \( m \text{SO}(X), m \text{PO}(X) \) and \( \alpha m(X) \), respectively.

(b) Let \((X, m_X)\) be an \( m \)-space and \( m \text{IT}(X) \) an \( m \)-iterate structure on \( X \). If \( m \text{IT}(X) = m \text{SO}(X) \) (resp. \( m \text{PO}(X), \alpha m(X), \beta m(X) \), \( m \text{BO}(X) \)), then we obtain the following definitions (for \( m \text{SO}(X), m \text{PO}(X) \) and \( \alpha m(X) \), they are provided in [19], [23] and [20], respectively):

\[
\begin{align*}
m \text{ITCl}(A) &= m \text{Cl}(A) \quad \text{(resp.} \ m \text{PO}(A), m \text{Cl}(A), \beta m \text{Cl}(A), m \text{BO}(A)), \\
m \text{ITInt}(A) &= m \text{Int}(A) \quad \text{(resp.} \ m \text{PO}(A), m \text{Int}(A), \beta m \text{Int}(A), m \text{BO}(A)).
\end{align*}
\]

Remark 6. (1) By Lemmas 1 and 3, we obtain Theorem 3.9 of [19], Theorems 2.3 and 2.4 of [23] and Theorems 3.8 and 3.9 of [20].

(b) By Lemma 2, we obtain Theorem 3.10 of [19], Lemma 3.9 of [22] and Theorem 3.10 of [20].

Definition 11. A function \( f : (X, m_X) \to (Y, m_Y) \) is said to be \( M \)-semi-continuous [19] (resp. \( M \)-precontinuous [22], \( \alpha M \)-continuous [20], \( \beta M \)-continuous, \( M \)-\( b \)-continuous) at \( x \in X \) if for each \( m \)-open set \( V \) containing \( f(x) \), there exists \( m \)-semiopen set (resp. \( m \)-preopen, \( \alpha m \)-open, \( \beta m \)-open, \( m \)-\( b \)-open).
$m$-b-open) set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. The function $f$ is said to be $M$-semicontinuous (resp. $M$-precontinuous, $\alpha M$-continuous, $\beta M$-continuous, $M$-b-continuous) if it has this property at each $x \in X$.

**Remark 7.** By Definition 11 and Remark 5, it follows that a function $f : (X, m_X) \to (Y, m_Y)$ is $M$-semicontinuous if a function $f : (X, mSO(X)) \to (Y, m_Y)$ is $M$-continuous.

**Definition 12.** A function $f : (X, m_X) \to (Y, m_Y)$ is said to be MIT-continuous at $x \in X$ (on $X$) if $f : (X, mIT(X)) \to (Y, m_Y)$ is $M$-continuous at $x \in X$ (on $X$).

**Remark 8.** Let $(X, m_X)$ be a minimal space. If $mIT(X) = mSO(X)$ (resp. $mPO(X)$, $\alpha m(X)$, $\beta m(X)$, $mBO(X)$) and $f : (X, m_X) \to (Y, m_Y)$ is MIT-continuous, then $f$ is $M$-semicontinuous (resp. $M$-precontinuous, $\alpha M$-continuous, $\beta M$-continuous, $M$-b-continuous).

Since $mIT(X)$ has property $\mathcal{B}$, by Theorems 1 and 2 and Corollary 1 we have the following theorems.

**Theorem 3.** For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:

(a) $f$ is MIT-continuous;
(b) $f^{-1}(V)$ is MIT-open for every $m$-open set $V$ of $Y$;
(c) $f^{-1}(F)$ is MIT-closed for every $m$-closed set $F$ of $Y$;
(d) $mITCl(f^{-1}(B)) \subseteq f^{-1}(mCl(B))$ for every subset $B$ of $Y$;
(e) $f(mITCl(A)) \subseteq mCl(f(A))$ for every subset $A$ of $X$;
(f) $f^{-1}(mInt(B)) \subseteq mITInt(f^{-1}(B))$ for every subset $B$ of $Y$.

For a function $f : (X, m_X) \to (Y, m_Y)$, we define $D_{MIT}(f)$ as follows:

$$D_{MIT}(f) = \{x \in X : f \text{ is not MIT-continuous at } x\}.$$ 

**Theorem 4.** For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties hold:

$$D_{MIT}(f) = \bigcup_{G \subseteq m_Y} \{f^{-1}(G) \cap mITInt(f^{-1}(G))\}$$
$$= \bigcup_{B \subseteq \mathcal{P}(Y)} \{f^{-1}(mInt(B)) \cap mITInt(f^{-1}(B))\}$$
$$= \bigcup_{B \subseteq \mathcal{P}(Y)} \{mITCl(f^{-1}(B)) - f^{-1}(mCl(B))\}$$
$$= \bigcup_{A \subseteq \mathcal{P}(X)} \{mITCl(A) - f^{-1}(mCl(f(A)))\}$$
$$= \bigcup_{F \subseteq \mathcal{F}} \{mITCl(f^{-1}(F)) - f^{-1}(F)\},$$

where $\mathcal{F}$ is the family of $m$-closed sets of $(Y, m_Y)$.

**Remark 9.** (a) If $mIT(X) = mSO(X)$ (resp. $mPO(X)$, $\alpha m(X)$, $\beta m(X)$, $mBO(X)$) and $f : (X, m_X) \to (Y, m_Y)$ is $MIT$-continuous, then by Theo-
rems 3 and 4 we obtain characterizations of $M$-semicontinuous (resp. $M$-pre-
continuous, $\alpha M$-continuous, $\beta M$-continuous, $M$-$b$-continuous) functions.

(b) If $\text{mIT}(X) = \text{mSO}(X)$ (resp. $\text{mPO}(X)$, $\text{mam}(X)$), then by Theorem 3
we obtain Theorem 3.15 of [19] (resp. Theorem 3.12 of [22], Theorem 3.14
of [20]).

For example, for $\text{mIT}(X) = \beta m(X)$ and $m_Y = \beta(Y)$, we obtain the
following characterizations.

**Corollary 2.** For a function $f : (X, m_X) \to (Y, m_Y)$, the following
properties are equivalent:

(a) $f$ is $\beta M$-continuous;
(b) $f^{-1}(V)$ is $\beta m$-open for every $\beta$-open set $V$ of $Y$;
(c) $f^{-1}(F)$ is $\beta m$-closed for every $\beta$-closed set $F$ of $Y$;
(d) $\beta m\text{Cl}(f^{-1}(B)) \subset f^{-1}(\beta \text{Cl}(B))$ for every subset $B$ of $Y$;
(e) $f(\beta m\text{Cl}(A)) \subset \beta \text{Cl}(f(A))$ for every subset $A$ of $X$;
(f) $f^{-1}(\beta \text{Int}(B)) \subset \beta m\text{Int}(f^{-1}(B))$ for every subset $B$ of $Y$.

5. Some properties of $\text{MIT}$-continuous functions

Since the study of $\text{MIT}$-continuity is reduced from the study of $M$-conti-
nuity, the properties of $\text{MIT}$-continuous functions follow from the properties
of $M$-continuous functions in [26].

**Definition 13.** An $m$-space $(X, m_X)$ is said to be $m$-$T_2$ [26] if for each
distinct points $x, y \in X$, there exist $U, V \in m_X$ containing $x$ and $y$, respec-
tively, such that $U \cap V = \emptyset$.

**Definition 14.** An $m$-space $(X, m_X)$ is said to be $\text{MIT}$-$T_2$ if the $m$-space
$(X, \text{mIT}(X))$ is $m$-$T_2$.

Hence, an $m$-space $(X, m_X)$ is $\text{MIT}$-$T_2$ if for each distinct points $x, y \in X$, there exist $U, V \in \text{mIT}(X)$ containing $x$ and $y$, respectively, such that $U \cap V = \emptyset$.

**Remark 10.** Let $(X, m_X)$ be an $m$-space. If $\text{mIT}(X) = \text{mSO}(X)$ (resp.
$\text{mPO}(X)$), then by Definition 14 we obtain the definition of $m$-semi-$T_2$
spaces in [21] (resp. $m$-pre-$T_2$-spaces in [23]).

**Lemma 4** (Popa and Noiri [26]). If $f : (X, m_X) \to (Y, m_Y)$ is an
$M$-continuous injection and $(Y, m_Y)$ is $m$-$T_2$, then $(X, m_X)$ is $m$-$T_2$.

**Theorem 5.** If $f : (X, m_X) \to (Y, m_Y)$ is an $\text{MIT}$-continuous injection
and $(Y, m_Y)$ is $m$-$T_2$, then $X$ is $\text{MIT}$-$T_2$.

**Proof.** The proof follows from Definition 14 and Lemma 4. □
Definition 15. An m-space \((X, m_X)\) is said to be m-compact [26] if every cover of \(X\) by \(m_X\)-open sets of \(X\) has a finite subcover.

A subset \(K\) of an m-space \((X, m_X)\) is said to be m-compact [26] if every cover of \(K\) by \(m_X\)-open sets of \(X\) has a finite subcover.

Definition 16. An m-space \((X, m_X)\) is said to be mIT-compact if the m-space \((X, m_{IT}(X))\) is m-compact.

A subset \(K\) of an m-space \((X, m_X)\) is said to be mIT-compact if every cover of \(K\) by \(m_{IT}\)-open sets of \(X\) has a finite subcover.

Remark 11. Let \((X, m_X)\) be an m-space. If \(m_{IT}(X) = m_{SO}(X)\) (resp. \(m_{PO}(X)\)), then by Definition 16 we obtain the definition of m-semicompact spaces in [21] (resp. m-precompact spaces in [23]).

Lemma 5 (Popa and Noiri [26]). Let \(f : (X, m_X) \to (Y, m_Y)\) be an \(M\)-continuous function. If \(K\) is an m-compact set of \(X\), then \(f(K)\) is m-compact.

Theorem 6. If \(f : (X, m_X) \to (Y, m_Y)\) is an MIT-continuous function and \(K\) is an mIT-compact set of \(X\), then \(f(K)\) is m-compact.

Proof. The proof follows from Definition 16 and Lemma 5.

Definition 17. A function \(f : (X, m_X) \to (Y, m_Y)\) is said to have a strongly m-closed graph (resp. m-closed graph) [26] if for each \((x, y)\) \(\in (X \times Y) \setminus G(f)\), there exist \(U \in m_X\) containing \(x\) and \(V \in m_Y\) containing \(y\) such that \([U \times mCl(V)] \cap G(f) = \emptyset\) (resp. \([U \times V] \cap G(f) = \emptyset\)).

Definition 18. A function \(f : (X, m_X) \to (Y, m_Y)\) is said to have a strongly mIT-closed graph (resp. mIT-closed graph) if a function \(f : (X, m_{IT}(X)) \to (Y, m_Y)\) has a strongly m-closed graph (resp. m-closed graph).

Hence, a function \(f : (X, m_X) \to (Y, m_Y)\) has a strongly mIT-closed graph (resp. mIT-closed graph) if for each \((x, y)\) \(\in (X \times Y) \setminus G(f)\), there exist \(U \in m_{IT}(X)\) containing \(x\) and \(V \in m_Y\) containing \(y\) such that \([U \times mCl(V)] \cap G(f) = \emptyset\) (resp. \([U \times V] \cap G(f) = \emptyset\)).

Lemma 6 (Popa and Noiri [26]). If \(f : (X, m_X) \to (Y, m_Y)\) is an \(M\)-continuous function and \((Y, m_Y)\) is m-T\(_2\), then \(f\) has a strongly m-closed graph.

Theorem 7. If \(f : (X, m_X) \to (Y, m_Y)\) is an MIT-continuous function and \((Y, m_Y)\) is m-T\(_2\), then \(f\) has a strongly mIT-closed graph.

Proof. The proof follows from Definition 18 and Lemma 6.
Lemma 7 (Popa and Noiri [26]). If $f : (X, m_X) \to (Y, m_Y)$ is a surjective function with a strongly $m$-closed graph, then $(Y, m_Y)$ is $m$-$T_2$.

Theorem 8. If $f : (X, m_X) \to (Y, m_Y)$ is a surjective function with a strongly $mIT$-closed graph, then $(Y, m_Y)$ is $m$-$T_2$.

Proof. The proof follows from Definition 18 and Lemma 7. ■

Lemma 8 (Popa and Noiri [26]). Let $(X, m_X)$ be an $m$-space and $m_X$ have property $B$. If $f : (X, m_X) \to (Y, m_Y)$ is an injective $M$-continuous function with an $m$-closed graph, then $X$ is $m$-$T_2$.

Theorem 9. If $f : (X, m_X) \to (Y, m_Y)$ is an injective $MIT$-continuous function with an $mIT$-closed graph, then $X$ is $mIT$-$T_2$.

Proof. The proof follows from Definition 18, Lemma 8 and the fact that $mIT(X)$ has property $B$. ■

Definition 19. An $m$-space $(X, m_X)$ is said to be $m$-connected [26] if $X$ cannot be written as the union of two nonempty sets of $m_X$.

Definition 20. An $m$-space $(X, m_X)$ is said to be $mIT$-connected if the $m$-space $(X, mIT(X))$ is $m$-connected.

Hence, the $m$-space $(X, mIT(X))$ is $m$-connected if $X$ cannot be written as the union of two nonempty sets of $mIT(X)$.

Lemma 9 (Popa and Noiri [26]). Let $f : (X, m_X) \to (Y, m_Y)$ be a function, where $m_X$ has property $B$. If $f$ is an $M$-continuous surjection and $(X, m_X)$ is $m$-connected, then $(Y, m_Y)$ is $m$-connected.

Theorem 10. If $f : (X, m_X) \to (Y, m_Y)$ is an $mIT$-continuous surjection and $(X, m_X)$ is $mIT$-connected, then $(Y, m_Y)$ is $m$-connected.

Proof. The proof follows from Definition 20, Lemma 9 and the fact that $mIT(X)$ has property $B$. ■

Definition 21. Let $(X, m_X)$ be an $m$-space and $A$ a subset of $X$. The $m$-frontier of $A$, $mFr(A)$, [27] is defined by $mFr(A) = mCl(A) \cap mCl(X \setminus A) = mCl(A) \setminus mInt(A)$.

Definition 22. Let $(X, m_X)$ be an $m$-space and $A$ a subset of $X$. The $mIT$-frontier of $A$, $mITFr(A)$, is defined by $mITFr(A) = mITCl(A) \cap mITCl(X \setminus A) = mITCl(A) \setminus mITInt(A)$. 
Lemma 10 (Popa and Noiri [28]). The set of all points of \( X \) at which a function \( f : (X, m_X) \rightarrow (Y, m_Y) \) is not \( M \)-continuous is identical with the union of the \( m \)-frontier of the inverse images of \( m \)-open sets of \( (Y, m_Y) \) containing \( f(x) \).

Theorem 11. The set of all points of \( X \) at which a function \( f : (X, m_X) \rightarrow (Y, m_Y) \) is not MIT-continuous is identical with the union of the \( m_{IT} \)-frontier of the inverse images of \( m \)-open sets of \( (Y, m_Y) \) containing \( f(x) \).

Proof. The proof follows from Definition 22 and Lemma 10. ■

References


