Abstract. In this paper, characterizations and properties of $e$-continuous functions are given. Moreover, Urysohn’s Lemma on $e$-normal spaces is proved.

Key words: $e$-open and $e$-closed subsets; $e$-continuous function; $e$-irresolute function; $e$-normal spaces; Urysohn’s lemma.

AMS Mathematics Subject Classification: 54A05, 54D15.

1. Introduction

In recent years, many researchers introduced different forms of continuous functions. El-Atik et al. [1] presented $\gamma$-open sets and $\gamma$-continuity. Hatir and Noiri et al. [5] has introduced $\delta$-$\beta$-open sets and $\delta$-$\beta$-continuity. Raychaudhurim and Mukherjee et al. [10] investigated $\delta$-preopen sets and $\delta$-semi-continuity but also discussed the relationship between $\delta$-$\beta$-continuity and $\delta$-semi-continuity. Noiri et al. [12] not only studied $\delta$-semi-sets and $\delta$-semi-continuity but also discussed the relationship between $\delta$-$\beta$-continuity and $\delta$-semi-continuity. In 2008, Ekici et al. [3] introduced the concept of $e$-open sets and investigated $e$-continuity. The purpose of this paper is to study further $e$-continuity. We will give characterizations and properties of $e$-continuity. We also discuss the relationship between $e$-continuity and other forms of continuity. In addition, Urysohn’s Lemma on $e$-normal spaces is proved.

2. Preliminaries

Throughout this paper, spaces always mean topological spaces with no separation properties assumed, and maps are onto. If $X$ is a space and $A \subset X$, then the interior and the closure of $A$ in $X$ are denoted by $iA, cA$, respectively.

Let $f_i : 2^X \rightarrow 2^X$ be a operator ($i = 1, 2, \ldots, n$) and $A \subset X$. We define

$$f_1f_2 \cdots f_n A = f_1(f_2(\cdots (f_n(A)) \cdots)).$$
Let $X$ be a space, $A \subset X$ and $x \in X$. $A$ is called regular open (resp. regular closed) if $A = icA$ (resp. $A = ciA$). $x$ is called a $\delta$-cluster point of $A$ if $A \cap icU \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure [7] of $A$ and is denoted by $c_\delta A$. $A$ is called $\delta$-closed if $c_\delta A = A$ and the complements are called $\delta$-open. The union of all $\delta$-open sets contained in $A$ is called the $\delta$-interior [7] of $A$ and is denoted by $i_\delta A$. Obviously, $A$ is $\delta$-open if and only if $A = i_\delta A$.

Let $(X, \tau)$ be a space and $x \in X$. Then $\tau(x)$ means the family of all open neighborhoods of $x$. Put

$$\tau_\delta = \{ A : A \text{ is } \delta\text{-open in } X \}.$$  

It is not difficult that $\tau_\delta$ forms a topology on $X$ and $\tau_\delta \subset \tau$.

**Definition 1.** Let $X$ be a space and $A \subset X$. Then $A$ is called

(a) $e$-open [3] if $A \subset ic_\delta A \cup ci_\delta A$.

(b) $\delta$-preopen [10] if $A \subset ic_\delta A$.

(c) $\delta$-semiopen [6] if $A \subset ci_\delta A$.

(d) $\delta$-$\beta$-open [4] if $A \subset ci_\delta A$.

(e) $b$-open [2] (or $\gamma$-open [1]) if $A \subset icA \cup ciA$.

The family of all $e$-open (resp. $\delta$-preopen, $\delta$-semiopen, $\delta$-$\beta$-open, $b$-open) subsets of $X$ is denoted by $EO(X)$ (resp. $\delta PO(X)$, $\delta SO(X)$, $\delta BO(X)$).

**Definition 2.** The complement of an $e$-open (resp. $\delta$-preopen, $\delta$-semiopen, $\delta$-$\beta$-open, $b$-open) set is called $e$-closed [3] (resp. $\delta$-preclosed [10], $\delta$-semiclosed [6], $\delta$-$\beta$-closed [4], $b$-closed [2]).

**Definition 3.** The union of all $e$-open (resp. $\delta$-preopen, $\delta$-semiopen, $\delta$-$\beta$-open, $b$-open) subsets of $X$ contained in $A$ is called the $e$-interior [3] (resp. $\delta$-preinterior [10], $\delta$-semi-interior [12], $\delta$-$\beta$-interior [4], $b$-interior [2]) of $A$ and is denoted by $i_e A$ (resp. $p_i_\delta A$, $s_i_\delta A$, $\beta_i_\delta A$, $i_b A$).

**Definition 4.** The intersection of all $e$-closed (resp. $\delta$-preclosed, $\delta$-semiclosed, $\delta$-$\beta$-closed, $b$-closed) sets of $X$ containing $A$ is called the $e$-closure [3] (resp. $\delta$-preclosure [10], $\delta$-semiclosure [12], $\delta$-$\beta$-closure [4], $b$-closure [2]) of $A$ and is denoted by $c_e A$ (resp. $p_c_\delta A$, $s_c_\delta A$, $\beta_c_\delta A$, $c_b A$).

**Lemma 1** ([4]). Let $X$ be a space and $A \subset X$. Then

(a) $p_i_\delta A = A \cap ic_\delta A$; $p_c_\delta A = A \cup ci_\delta A$.

(b) $s_i_\delta A = A \cap ci_\delta A$; $s_c_\delta A = A \cup ic_\delta A$.

(c) $\beta_i_\delta A = A \cap ci_\delta A$; $\beta_c_\delta A = A \cup ic_\delta A$.

**Proposition 1** ([3]). Let $X$ be a space and $A \subset X$. Then $A$ is $e$-open in $X$ if and only if $A = p_i_\delta A \cup s_i_\delta A$.  


Theorem 1 ([3]). Let $X$ be a space and $A \subset X$. Then
(a) $i_e A = A \cap (ic_\delta A \cup ci_\delta A)$.
(b) $c_e A = A \cup (ci_\delta A \cap ic_\delta A)$.
(c) $i_e (X - A) = X - c_e A$.
(d) $x \in i_e A$ if and only if $U \subset A$ for some $U \in EO(X)$ containing $x$.
(e) $A$ is $e$-open in $X$ if and only if $A = i_e A$.

Theorem 2 ([3]). Let $X$ be a space. Then
(a) The union of any family of $e$-open subsets of $X$ is $e$-open.
(b) The intersection of any family of $e$-closed subsets of $X$ is $e$-closed.

Proposition 2. Let $X$ be a space. Then the intersection of an open subset and a $e$-open subset is $e$-open in $X$.

Proof. Suppose $A \in EO(X)$ and $B \in \tau$. By Proposition 1, then $A \cap B = (p_1 \delta A \cup s_1 \delta A) \cap (p_2 \delta A \cap N B) \cup (s_2 \delta A \cap N B) = (p_1 \delta A \cap N B) \cup (s_2 \delta A \cap N B) \subset (p_2 \delta A \cap N B) \cup (s_2 \delta A \cap N B) = (A \cap s_2 \delta A \cap N B) \cup (A \cap p_2 \delta A \cap N B) \subset (ic_\delta A \cap ic_\delta B) \cup (ci_\delta A \cap ic_\delta B) = ic_\delta (A \cap B) \cup ci_\delta (A \cap B)$. Hence $A \cap B$ is $e$-open in $X$.

Definition 5. A function $f : X \to Y$ is called $\delta$-continuous [11] if $f^{-1}(V)$ is regular open in $X$ for each $V \in RO(Y)$.

Definition 6. A function $f : X \to Y$ is called $\delta$-$\beta$-continuous [5] (resp. $\gamma$-continuous [1], $\delta$-almost continuous [10], $\delta$-semi-continuous [12]) if $f^{-1}(V)$ is $\delta$-$\beta$-open (resp. $b$-open, $\delta$-preopen, $\delta$-semiopen) in $X$ for each open set $V$ in $Y$.

Lemma 2 ([9]). If $f : X \to Y$ is a function, $A \subset X$ and $B \subset Y$, then $f^{-1}(B) \subset A$ if and only if $B \subset Y - f(X - A)$.

3. e-continuous functions

Definition 7 ([3]). A function $f : (X, \tau) \to (Y, \sigma)$ is called $e$-continuous if $f^{-1}(V)$ is $e$-open in $X$ for each $V \in \sigma$.

Every $\delta$-almost continuous and $\delta$-semi-continuous is $e$-continuous but the converse is not true. Every $e$-continuous is $\delta$-$\beta$-continuous but the converse is also not true, as shown by the following Example 4.4 [3], Example 4.5 [3] and Example 1.

Example 1. Let $X = Y = \{x, y, z\}$, $\tau = \{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}$ and
\[
\sigma = \{\emptyset, \{x, z\}, Y\}.
\]

Let $f : X \to Y$ be the identity function.
Since $\tau(x) = \{\{x\}, \{x, y\}, X\}, \tau(y) = \{\{y\}, \{x, y\}, X\}$ and $\tau(z) = \{X\}$, then $c_\delta\{x, z\} = \{x, z\}$ and $i_\delta\{x, z\} = \{x, z\}$ and $ci_\delta\{x, z\} \cup ic_\delta\{x, z\} = \emptyset \cup \{x\} = \{x\}$. Therefore for each open subset $\{x, z\} \in \sigma$, then $f^{-1}\{\{x, z\}\} = \{x, z\} \subset ci_\delta f^{-1}\{\{x, z\}\} = \{x, z\}$ and $f^{-1}\{\{x, z\}\}$ is $\delta$-$\beta$-open in $X$. Hence $f$ is $\delta$-$\beta$-continuous.

But $f^{-1}\{\{x, z\}\} = \{x, z\} \not\subset ci_\delta f^{-1}\{\{x, z\}\} \cup ic_\delta f^{-1}\{\{x, z\}\} = \emptyset \cup \{x\} = \{x\}$ is not $e$-open in $X$. Hence $f$ is not $e$-continuous.

The following Theorem 3 gives some characterizations of $e$-continuity.

**Theorem 3.** Let $f : X \to Y$ be a function. Then the following are equivalent.

(a) $f$ is $e$-continuous;
(b) For each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists $U \in EO(X)$ containing $x$ such that $f(U) \subset V$;
(c) $f^{-1}(V)$ is $e$-closed in $X$ for each closed subset $V$ of $Y$;
(d) $ci_\delta f^{-1}(B) \cap ic_\delta f^{-1}(B) \subset f^{-1}(cB)$ for each $B \subset Y$;
(e) $f(ci_\delta A \cap ic_\delta A) \subset cf(A)$ for each $A \subset X$;
(f) $f^{-1}(i_B) \subset i_e f^{-1}(B)$ for each $B \subset Y$.

**Proof.** (a) $\iff$ (b), (a) $\iff$ (c) are obvious.

(c) $\Rightarrow$ (d). Let $B \subset Y$. By (3), then we obtain $f^{-1}(cB)$ is $e$-closed subset of $X$. Hence $ci_\delta f^{-1}(B) \cap ic_\delta f^{-1}(B) \subset ci_\delta f^{-1}(cB) \cap ic_\delta f^{-1}(cB) \subset f^{-1}(cB)$.

(d) $\Rightarrow$ (c). For any closed subset $V \subset Y$. By (4), then we have $ci_\delta f^{-1}(V) \cap ic_\delta f^{-1}(V) \subset f^{-1}(cV) = f^{-1}(V)$. Hence $f^{-1}(V)$ is $e$-closed in $X$.

(d) $\Rightarrow$ (e). Put $B = f(A)$. By (4), then we obtain $ci_\delta f^{-1}(f(A)) \cap ic_\delta f^{-1}(f(A)) \subset f^{-1}(cf(A))$ and $ci_\delta A \cap ic_\delta A \subset f^{-1}(cf(A))$. Hence $f(ci_\delta A \cap ic_\delta A) \subset cf(A)$.

(e) $\Rightarrow$ (d) is obvious.

(c) $\Rightarrow$ (f). Let $B \subset Y$, then $Y - iB$ is closed subset in $Y$. By (3), then we have $f^{-1}(Y - iB) \in EC(X)$ and $ci_\delta f^{-1}(Y - iB) \cap ic_\delta f^{-1}(Y - iB) \subset f^{-1}(Y - iB)$. Thus, we obtain $(X - (ci_\delta f^{-1}(iB))) \cap (X - (ic_\delta f^{-1}(iB))) \subset X - f^{-1}(iB)$ and $X - (ci_\delta f^{-1}(iB) \cup ic_\delta f^{-1}(iB)) \subset X - f^{-1}(iB)$. Hence $f^{-1}(iB) \subset ci_\delta f^{-1}(iB) \cup ic_\delta f^{-1}(iB) \subset ci_\delta f^{-1}(B) \cup ic_\delta f^{-1}(B)$ and $f^{-1}(iB) \subset i_e f^{-1}(B)$.

(f) $\Rightarrow$ (c) is obvious.$\blacksquare$

**Theorem 4.** Let $f : X \to Y$ be a function. If $if(A) \subset f(i_e A)$ for each $A \subset X$, then $f$ is $e$-continuous.

**Proof.** Suppose that $x \in X$ and $V$ is an open neighborhood of $f(x)$. Since $if(A) \subset f(i_e A)$, then $V = iV = if(f^{-1}(V)) \subset f(i_e f^{-1}(V))$. Thus,
we have \( f^{-1}(V) \subset i_e f^{-1}(V) \). Set \( U = f^{-1}(V) \), then \( U \in EO(X) \) containing \( x \) and \( f(U) \subset V \). By Theorem 3, then we obtain \( f \) is \( e \)-continuous. \( \blacksquare \)

### 4. Properties of \( e \)-continuous functions

**Theorem 5.** Let \( X \) and \( Y \) be two spaces and \( A \) be an open subset of \( X \). If \( f : X \to Y \) is \( e \)-continuous, then \( f|_A : A \to Y \) is also \( e \)-continuous.

**Proof.** Let \( V \) be open in \( Y \). Since \( f \) is \( e \)-continuous, then \( (f|_A)^{-1}(V) = (f|_A)^{-1}(V \cap f(A)) = f^{-1}(V \cap f(A)) = f^{-1}(V) \cap A \in EO(X) \). Therefore \( f|_A \) is \( e \)-continuous. \( \blacksquare \)

**Definition 8.** Let \( X \) be a space. Let \( \{x_\alpha, \alpha \in \Lambda\} \) be a net in \( X \) and \( x \in X \). Then \( \{x_\alpha, \alpha \in \Lambda\} \) is called \( e \)-converges to \( x \) in \( X \), we denote \( x_\alpha \to^e x \), if for every \( e \)-open set \( U \) containing \( x \) there exists a \( \alpha_0 \in \Lambda \) such that \( x_\alpha \in U \) for every \( \alpha \geq \alpha_0 \).

**Lemma 3.** Let \( X \) be a space and \( x \in X, A \subset X \). Then \( x \in c_eA \) if and only if there exists a net consisting of elements of \( A \) and converging to \( x \).

**Proof.** **Necessity.** Suppose \( x \in c_eA \) and we denote by \( U(x) \) the set of all \( e \)-open set containing \( x \) directed by the relation \( \supset \), i.e., define that \( U_1 \supset U_2 \) if \( U_1 \supset U_2 \). Thus, we can easily check that \( x_U \to^e x \) for each \( x_U \in U \cap A \).

**Sufficiency.** Let \( x_\alpha \to^e x \) in \( A \). For every \( e \)-open set \( U \) containing \( x \) there exists a \( \alpha_0 \in \Lambda \) such that \( x_\alpha \in U \) for every \( \alpha \geq \alpha_0 \). Thus, we have \( U \cap A \neq \emptyset \). Hence \( x \in c_eA \). \( \blacksquare \)

**Theorem 6.** A function \( f : X \to Y \) is \( e \)-continuous if and only if for any \( x \in X \), the net \( \{x_\alpha, \alpha \in \Lambda\} \) \( e \)-converges to \( x \) in \( X \), then the net \( \{f(x_\alpha), \alpha \in \Lambda\} \) converges to \( f(x) \) in \( Y \).

**Proof.** **Necessity.** Suppose a net \( \{x_\alpha, \alpha \in \Lambda\} \) \( e \)-converges to \( x \) in \( X \) and a open subset \( V \) of \( Y \) containing \( f(x) \). Then there exists a \( \alpha_0 \in \Lambda \) such that \( x_\alpha \in U \) for every \( \alpha \geq \alpha_0 \). Since \( f \) is \( e \)-continuous, then there exists a \( U \in EO(X) \) containing \( x \) such that \( f(U) \subset V \) with Theorem 3. Thus, we have \( f(x_\alpha) \in V \) for \( \alpha \geq \alpha_0 \). Hence \( \{f(x_\alpha), \alpha \in \Lambda\} \) converges to \( f(x) \) in \( Y \).

**Sufficiency.** By Theorem 3, we have \( f(c_eA) \subset cf(A) \). By Lemma 3, then there exists a net converging to \( x \) in \( A \) for every \( x \in c_eA \). By hypothesis, then there exists a net converges to \( f(x) \) in \( f(A) \). This implies the net \( e \)-converges to \( f(x) \). Again by Lemma 3, we obtain \( f(x) \in c_e f(A) \). Hence \( f \) is \( e \)-continuous. \( \blacksquare \)
Theorem 7. Let \( f, g : X \to Y \) be two functions and let \( h : X \to Y \times Y \) be a function, defined by \( h(x) = (f(x), g(x)) \) for each \( x \in X \). Then \( f \) and \( g \) are \( e \)-continuous if and only if \( h \) is \( e \)-continuous.

Proof. Necessity. Let a net \( \{x_{\alpha}, \alpha \in \Lambda \} \) \( e \)-converges to \( x \) for every \( x \in X \). For every open neighborhood \( W \) of \( h(x) \) there exist open subsets \( U \) and \( V \) in \( Y \) such that \( (f(x), g(x)) = h(x) \in U \times V \subset W \). Thus, we have \( f(x) \in U \) and \( g(x) \in V \). Since \( f \) is \( e \)-continuous, then there exists a \( \alpha_1 \in \Lambda \) such that \( f(x_{\alpha}) \in U \) for every \( \alpha \geq \alpha_1 \) with Theorem 6. Similarly, there exists a \( \alpha_2 \in \Lambda \) such that \( g(x_{\alpha}) \in V \) for every \( \alpha \geq \alpha_2 \). Set \( \alpha_0 = \max\{\alpha_1, \alpha_2\} \), then \( f(x_{\alpha}) \in U \) and \( g(x_{\alpha}) \in V \) for every \( \alpha \geq \alpha_0 \). Thus, we obtain \( h(x_{\alpha}) = (f(x_{\alpha}), g(x_{\alpha})) \in U \times V \subset W \). Hence \( h \) is \( e \)-continuous.

Sufficiency. Suppose \( p_Y : Y \times Y \to Y \) be the natural projections and \( f = p_Y \circ h \). Let \( U \) is an open subset of \( Y \). Then \( f^{-1}(V) = h^{-1}(p_Y^{-1}(V)) \). Since \( p_Y \) is continuous, then \( p_Y^{-1}(V) \) is open set in \( Y \times Y \). Since \( h \) is \( e \)-continuous, then \( h^{-1}(p_Y^{-1}(V)) \) is \( e \)-open set in \( X \). Hence \( f \) is \( e \)-continuous. Similarly, we can prove that \( g \) is \( e \)-continuous.

Definition 9. Let \( \mathcal{F} \) be a filter base in a space \( X \) and \( x \in X \). Then \( \mathcal{F} \) is called \( e \)-converges to \( x \), we denote \( \mathcal{F} \to^e x \), if for every \( e \)-open set \( U \) containing \( x \), there exists a \( F \in \mathcal{F} \) such that \( F \subset U \).

Theorem 8. A function \( f : X \to Y \) is \( e \)-continuous if and only if the filter base \( f(\mathcal{F}) = \{f(A) : A \in \mathcal{F}\} \) converges to \( f(x) \) in \( Y \) for every filter base \( \mathcal{F} \) \( e \)-converges to \( x \) in \( X \).

Proof. Necessity. Suppose \( x \in X \) and \( V \) be an open set containing \( f(x) \) in \( Y \). Since \( f \) be \( e \)-continuous, then there exists a \( U \in EO(X) \) containing \( x \) such that \( f(U) \subset V \) with Theorem 3. Let \( \mathcal{F} \to^e x \), then there exists a \( F \in \mathcal{F} \) such that \( F \subset U \) for every \( U \in EO(X) \) containing \( x \). Thus, we have \( f(x) \in f(F) \subset f(U) \subset V \) in \( Y \) for every \( f(F) \in f(\mathcal{F}) \). Hence filter base \( f(\mathcal{F}) \) converges to \( f(x) \).

Sufficiency. Suppose \( x \in X \) and \( V \) be an open set containing \( f(x) \) in \( Y \). Let filter base \( \mathcal{U}(x) \) be the set of all \( e \)-open set \( U \) containing \( x \) in \( X \), then \( \mathcal{U}(x) \to^e x \). By hypothesis, then \( f(\mathcal{U}(x)) \) converges to \( f(x) \). Thus, we have \( F \subset V \) for some a \( F \in f(\mathcal{U}(x)) \) and there exists a \( U \in \mathcal{U}(x) \) such that \( f(U) \subset V \). Hence \( f \) is \( e \)-continuous.

Theorem 9. If \( f : X \to Y \) is \( e \)-continuous and \( g : Y \to Z \) is continuous, then the composition \( g \circ f : X \to Z \) is \( e \)-continuous.

Proof. Suppose \( x \in X \) and \( V \) be an open neighborhood of \( g(f(x)) \). Since \( g \) is continuous, then there exists a \( g^{-1}(V) \) open in \( Y \) containing \( f(x) \). Since \( f \) is \( e \)-continuous, then there exists a \( U \in EO(X) \) containing \( x \) such
that \( f(U) \subset g^{-1}(V) \). Thus, we have \((g \circ f)(U) \subset (g \circ g^{-1})(V) \subset V\). Hence \(g \circ f\) is e-continuous.

**Definition 10.** A function \( f : X \rightarrow Y \) is called e-irresolute if \( f^{-1}(V) \in EO(X) \) for each \( V \in EO(Y) \).

**Definition 11.** A function \( f : X \rightarrow Y \) is called e-open if the image of every e-open subset is e-open.

Every e-irresolute function is e-continuous but the converse is not true, and e-irresolute and openness are not relate to each other, as shown by the following Example 2 and Example 3.

**Example 2.** Let \( X = Y = \{x, y, z\}, \tau = \emptyset, \{x\}, \{y\}, \{x, y\}, X \) and
\[
\sigma = \emptyset, \{x, y\}, Y. \]
Let \( f : X \rightarrow Y \) be the identity function.
Since \( \tau(x) = \{\{x\}, \{x, y\}, X\}, \tau(y) = \{\{y\}, \{x, y\}, X\} \) and \( \tau(z) = \{X\} \), then \( c_\delta\{x, y\} = \{X\} \) and \( i_\delta\{x, y\} = \emptyset \). Thus we have \( c_\delta\{x, y\} \cup i_\delta\{x, y\} = \{X\} \cup \emptyset = \{X\} \). Therefore for each open set \( \{x, y\} \in \sigma \), then \( f^{-1}(\{x, y\}) = \{x, y\} \subset i_\delta f^{-1}(\{x, y\}) \cup i_\delta f^{-1}(\{x, y\}) = \{X\} \) and \( f^{-1}(\{x, y\}) \) is e-open in \( X \). Hence \( f \) is e-continuous.

Since \( \sigma(x) = \sigma(y) = \{\{x, y\}, Y\} \) and \( \sigma(z) = \{Y\} \), then \( c_\delta\{x, z\} = \{Y\} \) and \( i_\delta\{x, z\} = \emptyset \). Therefore \( \{x, z\} \subset i_\delta c_\delta\{x, z\} \cup i_\delta c_\delta\{x, z\} = \{Y\} \) and \( \{x, z\} \) is e-open set in \( Y \). But \( f^{-1}(\{x, z\}) = \{x, z\} \not\subset c_\delta f^{-1}(\{x, z\}) \) and \( i_\delta f^{-1}(\{x, z\}) = \emptyset \cup \{x\} = \{x\} \) is not e-open in \( X \). Hence \( f \) is not e-irresolute.

**Example 3.** Let \( X = Y = \{x, y, z\}, \tau = \emptyset, \{x\}, \{x, z\}, X \) and
\[
\sigma = \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z\}, Y. \]
Let \( f : X \rightarrow Y \) be the identity function.
Since \( \tau(x) = \{\{x\}, \{x, z\}, X\}, \tau(y) = \{\{y\}, \{x, y\}, X\} \) and \( \tau(z) = \{x, z\}, X \), then \( c_\delta\{x, y\} = c_\delta\{y, z\} = c_\delta\{z\} = c_\delta\{y\} = \{X\} \) and \( i_\delta\{x, y\} = i_\delta\{y, z\} = i_\delta\{z\} = i_\delta\{y\} = \emptyset \). Thus we have \( c_\delta\{x, y\} \cup i_\delta\{x, y\} = \{X\} \cup \emptyset = \{X\} \), \( c_\delta\{y, z\} \cup i_\delta\{y, z\} = \{X\} \cup \emptyset = \{X\} \), \( c_\delta\{z\} \cup i_\delta\{z\} = \{X\} \cup \emptyset = \{X\} \). Hence \( EO(X) = \tau \cup \{\{x, y\}, \{y, z\}, \{y\}\} \).

Since \( \sigma(x) = \{\{x\}, \{x, y\}, Y\}, \sigma(y) = \{\{y\}, \{x, y\}, \{y, z\}, Y\} \) and \( \sigma(z) = \{\{y, z\}, Y\} \) then \( c_\delta\{x, z\} = \{Y\} \), \( c_\delta\{z\} = \{y, z\} \) and \( i_\delta\{x, z\} = i_\delta\{z\} = \emptyset \). Thus we have \( c_\delta\{x, z\} \cup i_\delta\{x, z\} = \{Y\} \cup \emptyset = \{Y\} \) and \( c_\delta\{z\} \cup i_\delta\{z\} = \{y, z\} \cup \emptyset = \{y, z\} \). Hence \( \{x, z\}, \{y, z\} \in EO(Y) \).
Because \( f(\{x\}) = \{x\} \in \sigma, f(\{y\}) = \{y\} \in \sigma, f(\{z\}) = \{z\} \in EO(Y), \)
\( f(\{x, y\}) = \{x, y\} \in \sigma, f(\{y, z\}) = \{y, z\} \in \sigma \) and \( f(\{x, z\}) = \{x, z\} \in EO(Y). \) Thus \( f \) is \( e \)-irresolute.

Let \( \{x, z\} \in \tau, \) then \( f(\{x, z\}) = \{x, z\} \notin \sigma. \) Hence \( f \) is not open.

From Example 1, Example 2, Example 3, Example 4.4 [3] and Example 4.5 [3], we have the following relationships:

\[ \begin{align*}
\delta\text{-almost continuity} & \quad \leftrightarrow \quad \delta\text{-semi-continuity} \\
\delta\text{-semi-continuity} & \quad \leftrightarrow \quad e\text{-continuity} \\
e\text{-continuity} & \quad \leftrightarrow \quad e\text{-irresolute} \\
e\text{-irresolute} & \quad \leftrightarrow \quad e\text{-open}
\end{align*} \]

**Theorem 10.** Let \( f : X \to Y \) be \( e \)-open and \( g : Y \to Z \) be a function. If \( g \circ f : X \to Z \) is \( e \)-continuous, then \( g \) is \( e \)-continuous.

**Proof.** Suppose \( B \) is open in \( Z. \) Since \( g \circ f \) is \( e \)-continuous, then \( (g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \) is \( e \)-open. Since \( f \) is \( e \)-open, then \( f(f^{-1}(g^{-1}(B))) = g^{-1}(B) \) is \( e \)-open. Hence \( g \) is \( e \)-continuous. \( \blacksquare \)

**Theorem 11.** Let \( f : X \to Y \) be \( e \)-open and \( g : Y \to Z \) be a function. If \( g \circ f : X \to Z \) is \( e \)-continuous, then \( g \) is \( e \)-continuous.

**Proof.** Suppose \( y \in Y \) and \( V \) is an open neighborhood of \( g(y). \) Then there exists a \( x \in X \) such that \( f(x) = y. \) Since \( g \circ f \) is \( e \)-continuous, then there exists a \( U \in EO(X) \) containing \( x \) such that \( g(f(U)) = (g \circ f)(U) \subset V. \) Since \( f \) is \( e \)-open, then \( f(U) \in EO(Y). \) Hence \( g \) is \( e \)-continuous. \( \blacksquare \)

Let \( \{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\} \) and \( \{(Y_\alpha, \sigma_\alpha) : \alpha \in \Lambda\} \) be two families of pairwise disjoint spaces, i.e., \( X_\alpha \cap X_\alpha' = Y_\alpha \cap Y_\alpha' = \emptyset \) for \( \alpha \neq \alpha' \) and let \( f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha) \) be a function for each \( \alpha \in \Lambda. \)

Denote the product space \( \prod_{\alpha \in \Lambda} \{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\} \) and \( \prod_{\alpha \in \Lambda} \{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\} \) by \( \prod X_\alpha \) and \( \prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha \) denote the product function defined by \( f(\{x_\alpha\}) = \{f(x_\alpha)\} \) for each \( \{x_\alpha\} \in \prod X_\alpha. \) Let \( P_\alpha : \prod X_\alpha \to X_\alpha \) and \( Q_\alpha : \prod Y_\alpha \to Y_\alpha \) be the natural projections.

**Theorem 12.** The product function \( \prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha \) is \( e \)-continuous if and only if \( f_\alpha : X_\alpha \to Y_\alpha \) is \( e \)-continuous for every \( \alpha \in \Lambda. \)
Proof. Denote $X = \prod_{\alpha \in \Lambda} X_\alpha, Y = \prod_{\alpha \in \Lambda} Y_\alpha$ and $f = \prod_{\alpha \in \Lambda} f_\alpha$.

Necessity. Suppose $f$ is $e$-continuous and $Q_\alpha$ is continuous for any $\alpha \in \Lambda$. By Theorem 10, then $f_\alpha \circ P_\alpha = Q_\alpha \circ f$ is $e$-continuous. Since $P_\alpha$ is continuous surjection, then $f_\alpha$ is $e$-continuous with Theorem 11.

Sufficiency. Let $x = \{x_\alpha\} \in X$ and $V$ be an open subset of $Y$ containing $f(x)$, then there exists a basic open set $\prod_{\alpha \in \Lambda} W_\alpha$ such that $f(x) \in \prod_{\alpha \in \Lambda} W_\alpha \subset V$ and $\prod_{\alpha \in \Lambda} W_\alpha = \prod_{i=1}^n W_{\alpha_i} \times \prod_{\alpha \notin \alpha_i} Y_\alpha$ where $W_\alpha$ be an open subset of $Y$ for each $\alpha \in \{\alpha_i : 1 < i < n\}$. Since $f_\alpha$ is $e$-continuous, then there exists an $e$-open set $U_{\alpha_i}$ such that $f_\alpha(U_{\alpha_i}) \in W_\alpha$ for each $x_{\alpha_i} \in X_{\alpha_i}$ and for each $W_\alpha$ be an open subset of $Y_\alpha$ containing $f(x_{\alpha_i})$. Put $U = \prod_{i \in \mathbb{N}} U_{\alpha_i} \times \prod_{\alpha \notin \alpha_i} X_\alpha$, then $U$ is $e$-open in $X$ and $f(x) \in f_\alpha(\{x_\alpha\}) \in f(U) \subset \prod_{i \in \mathbb{N}} f_\alpha(U_{\alpha_i}) \times \prod_{\alpha \notin \alpha_i} Y_\alpha$. Set $y = y \in \prod_{i \in \mathbb{N}} f_\alpha(U_{\alpha_i}) \times \prod_{\alpha \notin \alpha_i} Y_\alpha$, then there exists a $x^* \in U_{\alpha_i}$ such that $y_{\alpha_i} = f_\alpha(x^*_{\alpha_i})$ for every $\alpha \in \{\alpha_i : 1 < i < n\}$. If $\alpha \neq \alpha_i$, then there exists $y_\alpha \in Y_\alpha = f(X_\alpha)$ and $x^* \in X_\alpha$ such that $y_\alpha = f_\alpha(x^*_{\alpha_i})$. Thus, we have $\{y_\alpha\} = y \in \prod_{i \in \mathbb{N}} W_{\alpha_i} \times \prod_{\alpha \notin \alpha_i} Y_\alpha \subset f(U) \times Y \subset f(U) \subset V$.

Hence $f$ is $e$-continuous.

Denote the topological sum $(\bigcup_{\alpha \in \Lambda} X_\alpha, \tau)$ of $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ by $\bigoplus_{\alpha \in \Lambda} X_\alpha$ and the topological sum $(\bigcup_{\alpha \in \Lambda} Y_\alpha, \sigma)$ of $\{(Y_\alpha, \sigma_\alpha) : \alpha \in \Lambda\}$ by $\bigoplus_{\alpha \in \Lambda} Y_\alpha$, where

$$
\tau = \{A \subset X : A \cap X_\alpha \in \tau_\alpha \text{ for every } \alpha \in \Lambda\},
$$

and

$$
\sigma = \{B \subset Y : B \cap Y_\alpha \in \sigma_\alpha \text{ for every } \alpha \in \Lambda\},
$$

A function $\bigoplus_{\alpha \in \Lambda} f_\alpha : \bigoplus_{\alpha \in \Lambda} X_\alpha \to \bigoplus_{\alpha \in \Lambda} Y_\alpha$, called a sum function of $\{f_\alpha : \alpha \in \Lambda\}$, is defined as follows: for every $x \in \bigcup_{\alpha \in \Lambda} X_\alpha$,

$$(\bigoplus_{\alpha \in \Lambda} f_\alpha)(x) = f_\beta(x) \text{ if there exists unique } \beta \in \Lambda \text{ such that } x \in X_\beta.$$

Theorem 13. The sum function $\bigoplus_{\alpha \in \Lambda} f_\alpha : \bigoplus_{\alpha \in \Lambda} X_\alpha \to \bigoplus_{\alpha \in \Lambda} Y_\alpha$ is $e$-continuous if and only if $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha)$ is $e$-continuous for every $\alpha \in \Lambda$.

Proof. Denote $f = \bigoplus_{\alpha \in \Lambda} f_\alpha, X = \bigoplus_{\alpha \in \Lambda} X_\alpha, Y = \bigoplus_{\alpha \in \Lambda} Y_\alpha$. 
Necessity. Suppose \( f \) is \( e \)-continuous. Then \( f|_{X_\alpha} = f_\alpha \) is \( e \)-continuous with Theorem 5.

Sufficiency. Let \( V \) be an open subset of \( Y \). Then \( V \cap Y_\alpha \in \sigma_\alpha \) for every \( \alpha \in \bigwedge \). Let \( x \in f^{-1}(V) \cap X_\alpha \), then \( f(x) \in V \) and \( f(x) \in Y_\alpha \). This implies that \( f(x) \in f_\alpha(x) \). Thus, we have \( f_\alpha(x) \in V \) and \( f_\alpha(x) \in V \cap Y_\alpha \). Hence \( x \in f_\alpha^{-1}(V \cap Y_\alpha) \). Conversely, \( f_\alpha^{-1}(V \cap Y_\alpha) \subset f^{-1}(V) \cap X_\alpha \). Thus, we obtain \( f^{-1}(V) \cap X_\alpha = f_\alpha^{-1}(V \cap Y_\alpha) \) for every \( \alpha \in \bigwedge \). Since \( f_\alpha \) is \( e \)-continuous, then \( f^{-1}(V) \cap X_\alpha \) is \( e \)-open in \( X_\alpha \). Thus, we have \( f^{-1}(V) \) is \( e \)-open in \( X \). Hence \( f \) is \( e \)-continuous.

\[ \blacksquare \]

5. Separation axioms and graph properties

**Definition 12.** A space \( X \) is called

(a) Urysohn [8] if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist open subsets \( U \) and \( V \) such that \( x \in U \), \( y \in V \), and \( cU \cap cV = \emptyset \).

(b) \( e \)-T1 if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( e \)-open subsets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( y \notin U \) and \( x \notin V \).

(c) \( e \)-T2 if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( e \)-open subsets \( U \) and \( V \) such that \( x \in U \), \( y \in V \), and \( U \cap V = \emptyset \).

**Theorem 14.** Let \( f : X \rightarrow Y \) be a \( e \)-continuous injection. Then the following hold.

(a) If \( Y \) is a \( T_1 \)-space, then \( X \) is \( e \)-T1.

(b) If \( Y \) is a \( T_2 \)-space, then \( X \) is \( e \)-T2.

(c) If \( Y \) is Urysohn, then \( X \) is \( e \)-T2.

**Proof.** (a) Let \( x \) and \( y \) be any distinct points in \( X \). Since \( Y \) is a \( T_1 \)-space, then there exist open subsets \( U \) and \( V \) of \( Y \) such that \( f(x) \in U \), \( f(y) \notin U \) and \( f(x) \in V \), \( f(y) \notin V \). Since \( f \) is \( e \)-continuous, then \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( e \)-open in \( X \) such that \( x \in f^{-1}(U) \), \( y \notin f^{-1}(U) \) and \( x \notin f^{-1}(V) \), \( y \in f^{-1}(V) \). Hence \( X \) is \( e \)-T1.

(b) Let \( x \) and \( y \) be any distinct points in \( X \). Since \( Y \) is a \( T_2 \)-space, then there exist open subsets \( U \) and \( V \) containing \( f(x) \) and \( f(y) \) in \( Y \), respectively, such that \( U \cap V = \emptyset \). Since \( f \) is \( e \)-continuous, then there exist \( e \)-open subsets \( A \) and \( B \) containing \( x \) and \( y \), respectively, such that \( f(A) \subset U \) and \( f(B) \subset V \). This implies that \( A \cap B = \emptyset \). Hence \( X \) is \( e \)-T2.

(c) Let \( x \) and \( y \) be any distinct points in \( X \). Since \( Y \) is Urysohn, then there exist open subsets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \), \( f(y) \in V \) and \( cU \cap cV = \emptyset \). Since \( f \) is \( e \)-continuous, then there exist \( e \)-open subsets \( A \) and \( B \) containing \( x \) and \( y \), respectively, such that \( f(A) \subset U \subset cU \) and \( f(B) \subset V \subset cV \). This implies that \( A \cap B = \emptyset \). Hence \( X \) is \( e \)-T2. 

\[ \blacksquare \]
Theorem 15. Let \( f, g : X \to Y \) be two functions. If \( f \) is continuous, \( g \) is \( e \)-continuous and \( Y \) is \( e \)-\( T_2 \), then \( \{ x \in X : f(x) = g(x) \} \) is \( e \)-closed in \( X \).

Proof. Denote \( A = \{ x \in X : f(x) = g(x) \} \). Let \( x \in X - A \). Then \( f(x) \neq g(x) \). Since \( Y \) is an \( e \)-\( T_2 \) space, then there exist \( e \)-open subsets \( U \) and \( V \) containing \( f(x) \) and \( g(x) \) in \( Y \), respectively, such that \( U \cap V = \emptyset \). Since \( f \) is continuous and \( g \) is \( e \)-continuous, then \( f^{-1}(U) \) is \( e \)-open and \( g^{-1}(V) \) is \( e \)-open in \( X \). This implies that \( x \in f^{-1}(U) \) and \( x \in g^{-1}(V) \). Put \( W = f^{-1}(U) \cap g^{-1}(V) \), then \( W \) is \( e \)-open in \( X \) with Proposition 2. Thus, we have \( f(W) \cap g(W) \subset U \cap V = \emptyset \). This implies that \( W \cap A = \emptyset \) and \( x \in W \subset X - A \). Hence \( X - A \) is \( e \)-open and \( A \) is \( e \)-closed in \( X \).

Definition 13. A space \( X \) is called \( e \)-regular if for each \( e \)-closed subset \( F \) and each point \( x \notin F \), there exist disjoint open subsets \( U \) and \( V \) such that \( x \in U \) and \( F \subset V \).

Theorem 16. Let a function \( f : X \to Y \) be a \( e \)-irresolute surjection. If \( X \) is \( e \)-regular, then \( Y \) is \( e \)-regular.

Proof. Suppose \( y \in Y \) and \( F \) is \( e \)-closed in \( Y \) such that \( y \notin F \). Since \( f \) is \( e \)-irresolute surjection, then there exists a \( x \in X \) such that \( y = f(x) \) and \( f^{-1}(F) \) is \( e \)-closed in \( X \) such that \( x \notin f^{-1}(F) \). Since \( X \) is \( e \)-regular, then there exist disjoint open subsets \( U \) and \( V \) such that \( x \in U \) and \( f^{-1}(F) \subset V \). This implies \( y = f(x) \in f(U) \subset Y \setminus f(X - U) \). By Lemma 2, \( F \subset Y \setminus f(X - U) \). Note that \( Y \setminus f(X - U) \) and \( Y \setminus f(X - V) \) are disjoint open subsets of \( Y \). Hence \( Y \) is \( e \)-regular.

Definition 14. A space \( X \) is called \( e \)-normal if for every pair of disjoint \( e \)-closed subsets \( A \) and \( B \), there exist disjoint open subsets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

Theorem 17. Let a function \( f : X \to Y \) be \( e \)-irresolute. If \( X \) is \( e \)-normal, then \( Y \) is also \( e \)-normal.

Proof. Let \( A \) and \( B \) be disjoint \( e \)-closed subsets of \( Y \). Since \( f \) is \( e \)-irresolute, then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( e \)-closed subsets of \( X \). Since \( X \) is \( e \)-normal, then there exist disjoint open subsets \( U \) and \( V \) in \( X \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). By Lemma 2, \( A \subset Y \setminus f(X - U) \) and \( B \subset Y \setminus f(X - V) \). Note that \( Y \setminus f(X - U) \) and \( Y \setminus f(X - V) \) are disjoint open subsets of \( Y \). Hence \( Y \) is \( e \)-normal.

Lemma 4. A space \( X \) is \( e \)-normal if and only if for each \( e \)-closed subset \( F \) and \( e \)-open subset \( U \) containing \( F \), there exists an open set \( V \) such that \( F \subset V \subset c_eV \subset U \).
Then we obtain \( U \) is an -normal space, then there exist disjoint open subsets \( U_1, V_1 \) such that \( F \subset U_1 \) and \( X - U \subset V_1 \). This implies that \( X - V_1 \subset U \). Since \( U_1 \cap V_1 = \emptyset \), then we obtain \( c_e U_1 \subset X - V_1 \). Set \( V = U_1 \), then \( c_e U_1 \subset X - V_1 \subset U \). Therefore, \( F \subset V \subset c_e V \subset X - V_1 \subset U \).

**Sufficiency.** The proof is obvious.

Below we give Urysohn’s Lemma on -normal spaces.

**Theorem 18.** A space \( X \) is -normal if and only if for each pair of disjoint e-closed subsets \( A \) and \( B \) of \( X \), there exists a continuous map \( f : X \to [0, 1] \) such that \( f(A) = \{0\} \) and \( f(B) = \{1\} \).

**Proof.** **Sufficiency.** Suppose that for each pair of disjoint e-closed subsets \( A \) and \( B \) of \( X \), there exists a continuous map \( f : X \to [0, 1] \) such that \( f(A) = \{0\} \) and \( f(B) = \{1\} \). Put \( U = f^{-1}([^0, 1/2)), V = f^{-1}((1/2, 1]) \), then \( U \) and \( V \) are disjoint open subsets of \( X \) such that \( A \subset U \) and \( B \subset V \). Hence \( X \) is e-normal.

**Necessity.** Suppose \( X \) is e-normal. For each pair of disjoint e-closed subsets \( A \) and \( B \) of \( X \), \( A \subset X - B \), where \( A \) is e-closed in \( X \) and \( X - B \) is e-open in \( X \), by Lemma 4, there exists an open subset \( U_{1/2} \) of \( X \) such that

\[
A \subset U_{1/2} \subset c_e U_{1/2} \subset X - B.
\]

Since \( A \subset U_{1/2} \), \( A \) is e-closed in \( X \) and \( U_{1/2} \) is e-open in \( X \), then there exists an open subset \( U_{1/4} \) of \( X \) such that \( A \subset U_{1/4} \subset c_e U_{1/4} \subset U_{1/2} \) by Lemma 4. Since \( c_e U_{1/2} \subset X - B \), \( c_e U_{1/2} \) is e-closed in \( X \) and \( X - B \) is e-open in \( X \), then there exists an open subset \( U_{3/4} \) of \( X \) such that \( c_e U_{1/2} \subset U_{3/4} \subset c_e U_{3/4} \subset X - B \) by Lemma 4. Thus, there exist two open subsets \( U_{1/2} \) and \( U_{3/4} \) of \( X \) such that

\[
A \subset U_{1/4} \subset c_e U_{1/4} \subset U_{1/2} \subset c_e U_{1/2} \subset U_{3/4} \subset c_e U_{3/4} \subset X - B.
\]

We get a family \( \{U_{m/2^n} : 1 \leq m < 2^n, n \in N\} \) of open subsets of \( X \), denotes \( \{U_{m/2^n} : 1 \leq m < 2^n, n \in N\} \) by \( \{U_\alpha : \alpha \in \Gamma\} \). \( \{U_\alpha : \alpha \in \Gamma\} \) satisfies the following condition:

(a) \( A \subset U_\alpha \subset c_e U_\alpha \subset X - B \),
(b) if \( \alpha < \alpha' \), then \( c_e U_\alpha \subset U_{\alpha'} \).

We define \( f : X \to [0, 1] \) as follows:

\[
f(x) = \begin{cases} 
\inf\{\alpha \in \Gamma : x \in U_\alpha\}, & \text{if } x \in U_\alpha \text{ for some } \alpha \in \Gamma, \\
1, & \text{if } x \not\in U_\alpha \text{ for any } \alpha \in \Gamma.
\end{cases}
\]

For each \( x \in A \), \( x \in U_\alpha \) for any \( \alpha \in \Gamma \) by (1), so \( f(x) = \inf\{\alpha \in \Gamma : x \in U_\alpha\} = \inf\Gamma = 0 \). Thus, \( f(A) = \{0\} \).
For each $x \in B$, $x \not\in X - B$ implies $x \not\in U_\alpha$ for any $\alpha \in \Gamma$ by (1), so $f(x) = 1$. Thus, $f(B) = \{1\}$.

We have to show $f$ is continuous.

For $x \in X$ and $\alpha \in \Gamma$, we have the following Claim:

**Claim 1:** if $f(x) < \alpha$, then $x \in U_\alpha$.

Suppose $f(x) < \alpha$, then $\inf\{\alpha \in \Gamma : x \in U_\alpha\} < \alpha$, so there exists $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha_1 < \alpha$. By (2), $c_eU_{\alpha_1} \subset U_\alpha$. Notice that $x \in U_{\alpha_1}$. Hence $x \in U_\alpha$.

**Claim 2:** if $f(x) > \alpha$, then $x \not\in c_eU_\alpha$.

Suppose $f(x) > \alpha$, then there exists $\alpha_1 \in \Gamma$ such that $\alpha < \alpha_1 = f(x)$. Notice that $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ implies $\alpha_1 = \inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$.

Thus, $\alpha_1 \not\in \{\alpha \in \Gamma : x \in U_\alpha\}$. So $x \not\in U_{\alpha_1}$. By (2), $c_eU_{\alpha_1} \subset U_\alpha$. Hence $x \not\in c_eU_\alpha$.

**Claim 3:** if $x \not\in c_eU_\alpha$, then $f(x) \geq \alpha$.

Suppose $x \not\in c_eU_\alpha$, we claim that $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Otherwise, there exists $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha \geq \beta$. $x \not\in c_eU_\alpha$ implies $\alpha \not\in \{\alpha \in \Gamma : x \in U_\alpha\}$. So $\alpha \neq \beta$. Thus $\alpha > \beta$. By (2), $c_eU_{\beta} \subset U_\alpha$. So $x \not\in \beta$, contradiction. Therefore $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Hence $\alpha \leq \inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$.

For $x_0 \in X$, if $f(x_0) \in (0, 1)$, suppose $V$ is an open neighborhood of $f(x_0)$ in $[0, 1]$, then there exists $\epsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V \cap (0, 1)$. Pick $\alpha', \alpha'' \in \Gamma$ such that

$$0 < f(x_0) - \epsilon < \alpha' < f(x_0) < \alpha'' < f(x_0) + \epsilon < 1.$$ 

By Claim 1 and Claim 2, $x_0 \in U_{\alpha''}$, $x_0 \not\in c_eU_{\alpha'}$. Put $U = U_{\alpha''} - c_eU_{\alpha'}$, then $U$ is an open neighborhood of $x_0$ in $X$.

We will prove that $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$. if $y \in f(U)$, then $y = f(x)$ for some $x \in U$. $x \in U$ implies that $x \in U_{\alpha''}$ and $x \not\in c_eU_{\alpha'}$. Since $x \in U_{\alpha''}$, then $\alpha'' \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Thus, $\alpha'' = \inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$. Notice that $\alpha'' < f(x_0) + \epsilon$. Therefore $f(x) < f(x_0) + \epsilon$. Since $x \not\in c_eU_{\alpha'}$, then $f(x) \geq \alpha'$ by Claim 3. Notice that $f(x_0) - \epsilon < \alpha'$. Therefore $f(x) > f(x_0) - \epsilon$. Hence, $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Therefore, $f(U) \subset V$. This implies $f$ is continuous at $x_0$. If $f(x_0) = 0$, or 1, the proof that $f$ is continuous at $x_0$ is similar.

**Theorem 19.** Let $f : X \rightarrow Y$ be a function and $G : X \rightarrow X \times Y$ be the graph function of $f$, defined by $G(x) = (x, f(x))$ for each $x \in X$. Then $f$ is $e$-continuous if and only if $G$ is $e$-continuous.

**Proof.** Necessity. Let $x \in X$ and $V$ be an open subset in $X \times Y$ containing $G(x)$. Then there exist open subsets $U_1 \subset X$ and $W \subset Y$
such that \( G(x) = (x, f(x)) \subset U_1 \times W \subset V \). Since \( f \) is \( e \)-continuous, then there exists a \( U_2 \in EO(X) \) such that \( f(U_2) \subset W \). Set \( U = U_1 \cap U_2 \), then \( U \in EO(X) \) with Proposition 2. Thus, we have \( G(U) \subset V \). Hence \( G \) is \( e \)-continuous.

Sufficiency. Let \( x \in X \) and \( V \) be an open subset of \( Y \) containing \( f(x) \). Then \( X \times V \) is an open subset containing \( G(x) \). Since \( G \) is \( e \)-continuous, then there exists \( U \in EO(X) \) such that \( G(U) \subset X \times V \). Thus, we have \( f(U) \subset V \). Hence \( f \) is \( e \)-continuous.

\[ \text{Definition 15. A graph } G(f) \text{ of a function } f : X \to Y \text{ is called strongly } e \text{-closed if for each } (x, y) \in (X \times Y) \setminus G(f), \text{ there exists a } U \in EO(X) \text{ containing } x \text{ and an open subset } V \text{ of } Y \text{ containing } y \text{ such that } (U \times V) \cap G(f) = \emptyset. \]

\[ \text{Theorem 20. Let } f : X \to Y \text{ be } e \text{-continuous and } Y \text{ be } e-T_2. \text{ Then } G(f) \text{ is } e \text{-strongly closed.} \]

\[ \text{Proof. Let } (x, y) \in (X \times Y) \setminus G(f). \text{ Then } f(x) \neq y. \text{ Since } Y \text{ is } e-T_2, \text{ then there exist disjoint } e \text{-open subsets } V \text{ and } W \text{ of } Y \text{ such that } f(x) \in V \text{ and } y \in W. \text{ Since } f \text{ is } e \text{-continuous, then there exists a } U \in EO(X) \text{ such that } f(U) \subset V. \text{ Thus, we have } f(U) \cap (W) = \emptyset. \text{ Hence } (U \times W) \cap G(f) = \emptyset \text{ and } G(f) \text{ is strongly } e \text{-closed.} \]

\[ \text{Theorem 21. Let } f : X \to Y \text{ be a } e \text{-continuous and injective. If } G(f) \text{ is strongly } e \text{-closed, then } X \text{ is } e-T_2. \]

\[ \text{Proof. Let } x, y \in X \text{ such that } x \neq y. \text{ Since } f \text{ is injective, then } f(x) \neq f(y) \text{ and } (x, f(y)) \notin G(f). \text{ Since } G(f) \text{ is strongly } e \text{-closed, there exists a } U \in EO(X) \text{ and an open subset } W \text{ of } Y \text{ such that } (x, f(y)) \in U \times W \text{ and } (U \times W) \cap G(f) = \emptyset. \text{ Thus, we have } f(U) \cap W = \emptyset. \text{ Since } f \text{ is } e \text{-continuous, then there exists a } y \in V \in EO(X) \text{ such that } f(V) \subset W. \text{ This implies that } f(U) \cap f(V) = \emptyset. \text{ Hence } U \cap V = \emptyset \text{ and } X \text{ is } e-T_2. \]

6. \( e \)-connectedness and covering properties

\[ \text{Definition 16. A space } X \text{ is called } e \text{-connected if } X \text{ is not the union of two disjoint nonempty } e \text{-open subsets.} \]

\[ \text{Theorem 22. Let } f : X \to Y \text{ be } e \text{-continuous. If } X \text{ is } e \text{-connected, then } Y \text{ is connected.} \]

\[ \text{Proof. Suppose } Y \text{ is not a connected space. Then there exist nonempty disjoint open subsets } A \text{ and } B \text{ such that } Y = A \cup B. \text{ Since } f \text{ is } e \text{-continuous, then } f^{-1}(A) \text{ and } f^{-1}(B) \text{ are } e \text{-open subsets of } X. \text{ Thus, we obtain } f^{-1}(A) \]
and \( f^{-1}(B) \) are nonempty disjoint subsets and \( X = f^{-1}(A) \cup f^{-1}(B) \). This is contrary to the hypothesis that \( X \) is a \( e \)-connected space. Hence \( Y \) is connected.

**Corollary 1.** Let \( f : X \to Y \) be \( e \)-irresolute. If \( X \) is \( e \)-connected, then \( Y \) is \( e \)-connected.

**Definition 17.** A space \( X \) is called \( e \)-Lindelöf (resp. \( e \)-compact) if every \( e \)-open cover of \( X \) has a countable (resp. finite) subcover.

**Theorem 23.** Let \( f : X \to Y \) be \( e \)-continuous. If \( X \) is \( e \)-Lindelöf, then \( Y \) is Lindelöf.

**Proof.** Let \( \{ U_\alpha : \alpha \in \Lambda \} \) is an open cover of \( Y \). Since \( f \) is an \( e \)-continuous function, then \( f^{-1}(\{ U_\alpha : \alpha \in \Lambda \}) \) is an \( e \)-open cover of \( X \). Since \( X \) is \( e \)-Lindelöf, then there exists a countable subcover \( f^{-1}(\{ U_{\alpha i} : U_{\alpha i} \in \{ U_\alpha \}, 1 < i < \infty, \alpha \in \Lambda \}) \) in \( X \). Thus, we have \( \{ U_{\alpha i} : U_{\alpha i} \in \{ U_\alpha \}, 1 < i < \infty, \alpha \in \Lambda \} \) is a countable subcover of \( Y \). Hence \( Y \) is Lindelöf.

Similarly, we can prove the following Theorem 24.

**Theorem 24.** Let \( f : X \to Y \) be \( e \)-continuous. If \( X \) is \( e \)-compact, then \( Y \) is compact.

**Acknowledgement.** This paper is supported by the Innovation Project of Guangxi University for Nationalities (No. gxun-chx2011081).

**References**


**Tusheng Xie**  
**College of Mathematics and Information Science**  
**Guangxi University**  
**Nanning, Guangxi 530004, P.R. China**  
*e-mail: tushengxie@126.com*

**Haining Li**  
**College of Mathematics and Computer Science**  
**Guangxi University for Nationalities**  
**Nanning, Guangxi 530006, P.R. China**  
*e-mail: hning100@126.com*

Received on 07.11.2011 and, in revised form, on 23.03.2012.