Abstract. In this note, we prove some common fixed point theorems for occasionally weakly compatible mappings satisfying an implicit relation and a contractive modulus.

Key words: weakly compatible mappings, compatible mappings, occasionally weakly compatible mappings, implicit relations, contractive modulus, common fixed point theorems, metric space.

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1. Introduction

Let $S$ and $T$ be two self mappings of a metric space. Sessa [7] defined $S$ and $T$ to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all $x$ in $\mathcal{X}$. In 1986, Jungck [3] introduced the concept of compatibility as follows: $S$ and $T$ above are compatible if $\lim_{n \to \infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x$ for some $x \in \mathcal{X}$. Recently, in 2008, Al-Thagafi and Shahzad [1] weakened the above notion by giving the so-called occasionally weak compatibility. Let $\mathcal{X}$ be a set. $S$ and $T : \mathcal{X} \to \mathcal{X}$ are said to be occasionally weakly compatible if and only if, there is a point $x$ in $\mathcal{X}$ which is a coincidence point of $S$ and $T$ at which $S$ and $T$ commute.

Definition 1. A function $M : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $M(0) = 0$ and $M(t) < t$ for $t > 0$.

Theorem 1 ([2]). Let $\mathcal{X}$ be a set endowed with a symmetric $d$. Suppose $A, B, S$ and $T$ are four self mappings of $(\mathcal{X}, d)$ satisfying the conditions:

\[d^2(Ax, By) \leq \max\{M(d(Sx, Ty))M(d(Sx, Ax)), M(d(Sx, Ty))
\quad M(d(Ty, By)), M(d(Sx, Ax))M(d(Ty, By)),
\quad M(d(Sx, By))M(d(Ty, Ax))\},\]
for all \( x, y \in \mathcal{X} \), where \( M \) is contractive modulus, the pairs \((A, S)\) and \((B, T)\) are owc. Then \( A, B, S \) and \( T \) have a unique common fixed point.

In [6] and [5] is initiated the study of fixed point for mappings satisfying implicit relations. The purpose of this paper is to prove a general fixed point theorem for four mappings satisfying an implicit relation which generalizes Theorem 1.

2. Implicit relations

**Definition 2.** Let \((FM)\) be the set of all functions \( F(t_1, t_2, t_3, t_4, t_5, t_6) \) satisfying the following conditions:

\((Fm)\): \( F \) is increasing in variable \( t_1 \),

\((Fu)\): \( F(t, t, 0, 0, t, t) > 0 \) for every \( t > 0 \).

**Example 1.** \( F = t^2 - \max\{M(t_2)M(t_3), M(t_2)M(t_4), M(t_3)M(t_4), M(t_5)M(t_6)\} \), where \( M \) is a contractive modulus.

\((Fm)\): Obviously,

\((Fu)\): \( F(t, t, 0, 0, t, t) = t^2 - M^2(t) > 0 \) for every \( t > 0 \).

**Example 2.** \( F = t_1 - k \max\{M(t_2), M(t_3), M(t_4), \frac{M(t_5) + M(t_6)}{2}\} \), where \( M \) is a contractive modulus and \( k \in (0, 1) \).

\((Fm)\): Obviously,

\((Fu)\): \( F(t, t, 0, 0, t, t) = t - kM(t) > 0 \) for every \( t > 0 \).

**Example 3.**

\[
F = t^2_1 - k_1 \max\{M^2(t_2), M^2(t_3), M^2(t_4)\} - k_2 \max\{M(t_3)M(t_5), M(t_4)M(t_6)\} - k_3 M(t_5)M(t_6),
\]

where \( M \) is a contractive modulus, \( k_1 > 0, k_2, k_3 \geq 0 \), and \( k_1 + k_3 \leq 1 \).

\((Fm)\): Obviously,

\((Fu)\): \( F(t, t, 0, 0, t, t) = t^2 - (k_1 + k_3)M^2(t) > 0 \ \forall t > 0 \).

**Example 4.** \( F = t^2_1 - aM^2(t_2) - \frac{bM(t_5)M(t_6)}{1 + M^2(t_3) + M^2(t_4)} \), where \( M \) is a contractive modulus, \( a > 0, b \geq 0 \), and \( a + b \leq 1 \).

\((Fm)\): Obviously,

\((Fu)\): \( F(t, t, 0, 0, t, t) = t^2 - (a + b)M^2(t) > 0 \ \forall t > 0 \).

**Example 5.** \( F = t_1 - \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\} \), where \( M \) is a contractive modulus.

\((Fm)\): Obviously,

\((Fu)\): \( F(t, t, 0, 0, t, t) = t - M(t) > 0 \ \forall t > 0 \).
Lemma 1 (Jungck and Rhoades [4]). Let $\mathcal{X}$ be a nonempty set and $S$ and $T$ be occasionally weakly compatible self mappings on $\mathcal{X}$. If $S$ and $T$ have a unique common point of coincidence $w = Sx = Tx$, then $w$ is the unique common fixed point of $S$ and $T$.

3. Common fixed point theorems

Theorem 2. Let $(\mathcal{X},d)$ be a metric space and $A$, $B$, $S$, $T : (\mathcal{X},d) \to (\mathcal{X},d)$ such that

$$
F(d(Ax,By), M(d(Sx,Ty)), M(d(Sx,Ax)),
M(d(Ty,By)), M(d(Sx,By)), M(d(Ty,Ax))) \leq 0,
$$

for all $x,y \in \mathcal{X}$, where $F$ satisfies property $(Fu)$ and $M$ is a contractive modulus. If there exist $x$, $y$ in $\mathcal{X}$ such that $Ax = Sx$ and $By = Ty$, then, $A$ and $S$ have a unique point of coincidence and $B$ and $T$ have a unique point of coincidence (resp. $u = Ax = Sx$ and $v = By = Ty$). Moreover $u = v$.

Proof. First we prove that $Ax = By$. Suppose contrary. By (2) we obtain

$$
F(d(Ax,By), M(d(Ax,By)), M(d(Ax,Ax)),
M(d(By,By)), M(d(Ax,By)), M(d(By,Ax))) \leq 0.
$$

As $F$ is increasing in variable $t_1$, we have

$$
F(M(d(Ax,By)), M(d(Ax,By)), 0, 0,
M(d(Ax,By)), M(d(Ax,By))) \leq 0,
$$

a contradiction of $(Fu)$. Hence, $M(d(Ax,By)) = 0$ which implies $Ax = By = Sx = Ty = u = v$.

If there exists another point of coincidence for $A$ and $S$, $w = Az = Sz$ with $Az$ is distinct of $Ax$, then by (2) we have

$$
F(d(Az,By), M(d(Az,By)), M(d(Az,Az)),
M(d(By,By)), M(d(Az,By)), M(d(By,Az))) \leq 0.
$$

By condition $(Fm)$, we get

$$
F(M(d(Az,By)), M(d(Az,By)), 0, 0,
M(d(Az,By)), M(d(Az,By))) \leq 0,
$$

a contradiction of $(Fu)$. Hence $u$ is the unique point of coincidence for $A$ and $S$. Similarly, $v$ is the unique point of coincidence of $T$ and $B$ and $u = v$. Therefore $u = v$ is the unique point of coincidence for $A$ and $S$ and $B$ and $T$. ■
Theorem 3. Let $A, B, S, T$ self mappings of a metric space $(X, d)$ satisfying inequality (2) for all $x, y$ in $X$ where $F$ is in $(FM)$ and $M$ is a contractive modulus. If the pairs $(A, S)$ and $(B, T)$ are occasionally weakly compatible then, $A, B, S$ and $T$ have a unique common fixed point.

Proof. Since the pairs $(A, S)$ and $(B, T)$ are occasionally weakly compatible, then, $A$ and $S$ have a point of coincidence $u = Ax = Sx$ and $B$ and $T$ have a point of coincidence $v = By = Ty$. By the above theorem, $u = v$ and it is a unique common point of coincidence for $A$ and $S$ and for $B$ and $T$. By the above lemma $A$ and $S$ have $u$ as unique common fixed point and $B$ and $T$ have $u$ as the unique common fixed point. Therefore, $u$ is the unique common fixed point of $A, B, S$ and $T$. ■

Example 6. Let $X = [0, \infty[$ with the metric $d(x, y) = |x - y|$. Define

$$Ax = Bx = \begin{cases} \frac{3}{4} & \text{if } x \in [0, 1] \\ 1 & \text{if } x \in [1, \infty[ \end{cases}, \quad Sx = \begin{cases} 2 & \text{if } x \in [0, 1] \\ \frac{1}{x^2} & \text{if } x \in [1, \infty[ \end{cases},$$

and

$$Tx = \begin{cases} 2 & \text{if } x \in [0, 1] \\ \frac{1}{x} & \text{if } x \in [1, \infty[ \end{cases}.$$

First it is clear to see that $A$ and $S$ are occasionally weakly compatible as well as $B$ and $T$.

Take $M(t) = \frac{1}{2} t$ and

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\},$$

we get

(a) For $x, y \in [0, 1]$, we have $Ax = By = \frac{3}{4}$, $Sx = Ty = 2$ and

$$F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)))$$

$$= F(0, M(0), M(\frac{5}{4}), M(\frac{5}{4}), M(\frac{5}{4}), M(\frac{5}{4}))$$

$$= F(0, 0, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8})$$

$$= 0 - \max\{M(0), M(\frac{5}{8})\} = - \max\{0, \frac{5}{16}\} = - \frac{5}{16} \leq 0$$

because that $d(Ax, By) = 0$ and $\max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\} = \frac{5}{16}$ then $d(Ax, By) \leq \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}$. 

(b) For \( x, y \in [1, \infty[ \), we have \(Ax = By = 1, Sx = \frac{1}{x^2}, Ty = \frac{1}{y}\) and

\[
F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)), M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax)))
\]

\[
= F(0, M(\left|\frac{1}{x^2} - \frac{1}{y}\right|), M(\left|\frac{1}{x^2} - 1\right|), M(\left|\frac{1}{y} - 1\right|), M(\left|\frac{1}{x^2} - 1\right|), M(\left|\frac{1}{y} - 1\right|))
\]

\[
= F(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{y} - 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{y} - 1)
\]

\[
= 0 - \max\{M(\left|\frac{1}{x^2} - \frac{1}{y}\right|), M(\left|\frac{1}{x^2} - 1\right|), M(\left|\frac{1}{y} - 1\right|)
\]

\[
= -\max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} \leq 0
\]

because that \(d(Ax, By) = 0\) and \(\max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\} \leq 1\) then \(d(Ax, By) \leq \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}\).

(c) For \( x \in [0, 1[ , y \in [1, \infty[ \), we have \(Ax = \frac{3}{4}, By = 1, Sx = 2, Ty = \frac{1}{y}\) and

\[
F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)), M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax)))
\]

\[
= F\left(\frac{1}{4}, M(\left|2 - \frac{1}{y}\right|), M(\left|2 - \frac{3}{4}\right|), M(\left|\frac{1}{y} - 1\right|), M(\left|2 - 1\right|), M(\left|\frac{1}{y} - \frac{3}{4}\right|)\right)
\]

\[
= F\left(\frac{1}{4}, M(\left|2 - \frac{1}{y}\right|), M(\left|\frac{5}{4}\right|), M(\left|\frac{1}{y} - 1\right|), M(1), M(\left|\frac{1}{y} - \frac{3}{4}\right|)\right)
\]

\[
= F\left(\frac{1}{4}, \frac{2 - \frac{1}{y}}{2}, \frac{5}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{y} - \frac{3}{4}\right)
\]

\[
= \frac{1}{4} - \max\{M(\left|\frac{2 - \frac{1}{y}}{2}\right|), M(\frac{5}{8}), M(\left|\frac{1}{y} - \frac{1}{2}\right|), M(\frac{1}{2}), M(\left|\frac{1}{y} - \frac{3}{4}\right|)\}
\]

\[
= \frac{1}{4} - \max\{\frac{2 - \frac{1}{y}}{4}, \frac{5}{16}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} \leq 0
\]
because that \(d(Ax, By) = \frac{1}{4}\) and \(M(d(Sx, By)) = \frac{1}{2}\) then

\[
d(Ax, By) \leq \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}.
\]

(d) Finally, for \(x \in [1, \infty[, y \in [0, 1[,\) we have \(Ax = 1, By = \frac{3}{4}, Sx = \frac{1}{x^2}, Ty = 2\) and

\[
F(d(Ax, By), M(d(Sx, Ty)), M(d(Sx, Ax)),
M(d(Ty, By)), M(d(Sx, By)), M(d(Ty, Ax)))
= F\left(\frac{1}{4}, M\left(\frac{1}{x^2} - 2\right), M\left(\frac{1}{x^2} - 1\right), M(2 - \frac{3}{4}), M\left(\frac{1}{x^2} - \frac{3}{4}\right), M(2 - 1)\right)
= F\left(\frac{1}{4}, \frac{|\frac{1}{x^2} - 2|}{2}, \frac{|\frac{1}{x^2} - 1|}{2}, 5, \frac{|\frac{1}{x^2} - \frac{3}{4}|}{2}, 1\right)
= \frac{1}{4} - \max\{M\left(\frac{1}{x^2} - 2\right), M\left(\frac{1}{x^2} - 1\right), M\left(\frac{5}{8}\right), M\left(\frac{1}{x^2} - \frac{3}{4}\right), M\left(\frac{1}{2}\right)\}
= \frac{1}{4} - \max\left\{\frac{1}{4}, \frac{5}{16}, \frac{1}{4}\right\} \leq 0
\]

because that \(d(Ax, By) = \frac{1}{4}\) and \(M(d(Ty, Ax)) = \frac{1}{4}\) then

\[
d(Ax, By) \leq \max\{M(t_2), M(t_3), M(t_4), M(t_5), M(t_6)\}.
\]

So, all the hypotheses of the above theorem are satisfied and 1 is the unique common fixed point of mappings \(A, B, S\) and \(T\).

**Corollary 1.** Theorem 1.

**Proof.** The proof follows by Theorem 3 and Example 1. □

If \(A = B\) and \(S = T\) by Theorem 3 we obtain:

**Theorem 4.** Let \(A\) and \(S\) be self mappings of a metric space \((X, d)\) satisfying the inequality

\[
F(d(Ax, Ay), M(d(Sx, Sy)), M(d(Sx, Ax)),
M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))) \leq 0,
\]

for all \(x, y \in X\), where \(F\) is in \(F(M)\) and \(M\) is a contractive modulus. If \(A\) and \(S\) are occasionally weakly compatible, then \(A\) and \(S\) have a unique common fixed point.
Corollary 2. Let $A$ and $S$ be self mappings of a metric space $(X,d)$ satisfying the inequality
\[
d(Ax, Ay) \leq \max\{M(d(Sx, Sy)), M(d(Sx, Ax)), M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))\}
\]
for all $x, y \in X$. If $A$ and $S$ are occasionally weakly compatible, then $A$ and $S$ have a unique common fixed point.

Proof. The proof follows by Theorem 4 and Example 5.

Example 7. Let $X = [1, \infty)$, $Ax = x$, $Sx = 2x - 1$, $Mx = \frac{1}{2}x$ and $d(x, y) = |x - y|$. It follows that $AS(1) = SA(1) = 1$. Hence $A$ and $S$ are owc. On the other hand $d(Ax, Ay) = |x - y|, M(d(Sx, Sy)) = \frac{1}{2}d(Sx, Sy) = |x - y|$. Therefore
\[
d(Ax, Ay) \leq \max\{M(d(Sx, Sy)), M(d(Sx, Ax)), M(d(Sy, Ay)), M(d(Sx, Ay)), M(d(Sy, Ax))\}
\]
by Theorem 4, $A$ and $S$ have a unique common fixed point which is $x = 1$ because $A(1) = S(1) = 1$.

References


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