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∧\text{-}sets and ∨\text{-}sets in weak structures space due to Császár

Abstract. In this paper we introduce the concepts of ∧\text{-}sets and ∨\text{-}sets in a weak structure space due to Császár. It is shown that many results in previous papers can be considered as special cases of our results.

Key words: weak structure, ∧\text{-}set, ∨\text{-}set, w-T_1, w-T_{1/2}.

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1. Introduction

The notion of ∧\text{-}sets was introduced by Maki [7] in 1986. A subset A of a topological space is called a ∧\text{-}set if it is the intersection of all open sets containing A. Recently many authors have introduced and studied modifications of ∧\text{-}sets. By using a minimal structure, Cammaroto and Noiri [3] introduced the notions of ∧\text{-}m\text{-}sets and ∨\text{-}m\text{-}sets as unified forms of these modifications. Furthermore, recently Ekici and Roy [6] have introduced and investigated the notions of ∧\text{-}µ\text{-}sets and ∨\text{-}µ\text{-}sets on a generalized topological space (X,µ) due to Császár [4]. Quite recently, Császár [5] has introduced the notion of weak structures and obtained several fundamental properties of weak structures, moreover see [8].

In this paper, we introduce the notions of ∧\text{-}w\text{-}sets and ∨\text{-}w\text{-}sets on a weak structure space (X, w) and investigate the properties of sets and spaces related to ∧\text{-}w\text{-}sets and ∨\text{-}w\text{-}sets.

2. Preliminaries

Let X be a nonempty set and \mathcal{P}(X) the power set of X. A subfamily w of \mathcal{P}(X) is called a weak structure (briefly WS) [5] if \phi \in w. The pair (X, w) is called a weak structure (WS) space. Each member of a WS w is said to be w-open [5] and the complement of a w-open set is said to be w-closed. Let A be a subset of X. The union of all w-open sets contained in A is called the w-interior of A and is denoted by i_w(A) [5]. The intersection
of all \( w \)-closed sets containing \( A \) is called the \( w \)-closure of \( A \) and is denoted by \( c_w(A) \).

For the \( w \)-interior and the \( w \)-closure, the following lemmas are useful in the sequel.

**Lemma 1 (\([5]\)).** Let \( w \) be a WS on \( X \) and \( A, B \) subsets of \( X \), then

1. \( i_w(A) \subseteq A \subseteq c_w(A) \).
2. If \( A \subseteq B \) implies that \( i_w(A) \subseteq i_w(B) \) and \( c_w(A) \subseteq c_w(B) \).
3. \( i_w(i_w(A)) = i_w(A) \) and \( c_w(c_w(A)) = c_w(A) \).
4. \( i_w(X - A) = X - c_w(A) \) and \( c_w(X - A) = X - i_w(A) \).

**Lemma 2 (\([5]\)).** Let \( w \) be a WS on \( X \), then

1. \( x \in i_w(A) \) if and only if there exists \( W \in w \) such that \( x \in W \subseteq A \).
2. \( x \in c_w(A) \) if and only if \( W \cap A \neq \emptyset \) whenever \( x \in W \in w \).
3. If \( A \in w \), then \( A = i_w(A) \) and if \( A \) is \( w \)-closed, then \( A = c_w(A) \).

**Remark 1.** If \( w \) is a WS on \( X \), then

1. \( i_w(\emptyset) = \emptyset \) and \( c_w(X) = X \).
2. \( i_w(X) \) is the union of all \( w \)-open sets in \( X \).
3. \( c_w(\emptyset) \) is the intersection of all \( w \)-closed sets in \( X \).

We call a class \( \mu \subseteq \mathcal{P}(X) \) a generalized topology \([4]\) (briefly, GT) if \( \emptyset \in \mu \) and the arbitrary union of elements of \( \mu \) belongs to \( \mu \). A set \( X \) with a GT \( \mu \) on it is called a generalized topological space (briefly, GTS) and is denoted by \((X, \mu)\).

### 3. \( \wedge_w \)-sets and \( \vee_w \)-sets

**Definition 1.** Let \( w \) be a WS on a set \( X \) and \( A \subseteq X \). Then the subsets \( \wedge_w(A) \) and \( \vee_w(A) \) are defined as follows:

\[
\wedge_w(A) = \begin{cases} 
\cap \{G : A \subseteq G, G \in w\}, & \text{if there exists } G \in w \\
X, & \text{otherwise}
\end{cases}
\]

and

\[
\vee_w(A) = \begin{cases} 
\cup \{H : H \subseteq A, X - H \in w\}, & \text{if there exists } H \text{ such that } X - H \in w \text{ and } H \subseteq A; \\
\emptyset, & \text{otherwise}
\end{cases}
\]

**Proposition 1.** Let \( A, B \) and \( \{C_\alpha : \alpha \in \Delta\} \) be subsets of a WS on \( X \). Then the following properties hold:

1. \( B \subseteq \wedge_w(B) \).
(2) If \(A \subseteq B\), then \(\wedge_w(A) \subseteq \wedge_w(B)\).
(3) \(\wedge_w(\wedge_w(B)) = \wedge_w(B)\).
(4) \(\bigcup_{\alpha \in \Delta} (\wedge_w(C_\alpha)) \subseteq \wedge_w(\bigcup_{\alpha \in \Delta} C_\alpha)\).
(5) \(\wedge_w(\bigcap_{\alpha \in \Delta} C_\alpha) \subseteq \cap_{\alpha \in \Delta} (\wedge_w(C_\alpha))\).
(6) If \(A \in w\), then \(A = \wedge_w(A)\).
(7) \(\wedge_w(X - B) = X - \vee_w(B)\).
(8) \(\vee_w(B) \subseteq B\).
(9) If \(X - B \in w\), then \(B = \vee_w(B)\).
(10) If \(A \subseteq B\), then \(\vee_w(A) \subseteq \vee_w(B)\).
(11) \(\vee_w(\bigcup_{\alpha \in \Delta} C_\alpha) \supseteq \bigcup_{\alpha \in \Delta} (\wedge_w(C_\alpha))\).

**Proof.** (1), (6) and (8) are clear.

(2) If there does not exist any \(U \in w\) such that \(B \subseteq U\) then the proof is trivial. Suppose there exist \(V \in w\) such that \(B \subseteq V\) and that \(x \notin \wedge_w(B)\).
Then there exist a subset \(U \in w\) such that \(B \subseteq U\) with \(x \notin U\). Since \(A \subseteq B\), then \(x \notin \wedge_w(A)\) and thus \(\wedge_w(A) \subseteq \wedge_w(B)\).

(3) By (1), we have \(\wedge_w(\wedge_w(B)) \supseteq \wedge_w(B)\). Suppose that \(x \notin \wedge_w(B)\).
Then there exists \(U \in w\) such that \(B \subseteq U\) and \(x \notin U\). Since \(B \subseteq \wedge_w(B) \subseteq U\), we have \(x \notin \wedge_w(\wedge_w(B))\) and hence \(\wedge_w(\wedge_w(B)) \subseteq \wedge_w(B)\).

(4) The proof follows from (2).

(5) Suppose that \(x \notin \bigcap_{\alpha \in \Delta} (\wedge_w(C_\alpha))\). There exists \(\alpha_0 \in \Delta\) such that \(x \notin \wedge_w(C_{\alpha_0})\) and there exists a \(w\)-open set \(U\) such that \(x \notin U\) and \(C_{\alpha_0} \subseteq U\).
Since \(\bigcap_{\alpha \in \Delta} C_{\alpha_0} \subseteq C_{\alpha_0}\) we have \(x \notin \wedge_w(\bigcap_{\alpha \in \Delta} C_\alpha)\) and hence \(\wedge_w(\bigcap_{\alpha \in \Delta} C_\alpha) \subseteq \bigcap_{\alpha \in \Delta} (\wedge_w(C_\alpha))\).

(7) \(X - \vee_w(B) = \cap\{X - F : X - B \subseteq X - F, X - F \in w\} = \wedge_w(X - B)\).
(9) If \(X - B \in w\), then by (6) and (7) \(X - B = \wedge_w(X - B) = X - \vee_w(B)\).
Hence \(B = \vee_w(B)\).

(10) This follows from (2) and (7).
(11) This follows from (10).

In (4), (5) and (11) of Proposition 1, the equality does not necessarily hold as shown in the next example.

**Example 1.** (1) Let \(X = \{a, b, c\}\). Consider the WS \(w=\{\phi, \{a\}, \{b\}\}\) on \(X\). Let \(A = \{a, b\}\) and \(B = \{a, c\}\). Then \(\wedge_w(A) = X\), \(\wedge_w(B) = X\) and \(\wedge_w(A \cap B) = \{a, b\}\). Thus \(\wedge_w(A \cap B) \neq \wedge_w(A) \cap \wedge_w(B)\).

(2) Let \(X = \{a, b, c\}\). Consider the WS \(w = \{\phi, \{a\}, \{b\}\}\) on \(X\). Let \(A = \{a\}\) and \(B = \{b\}\). Then \(\wedge_w(A) = \{a\}\), \(\wedge_w(B) = \{b\}\) and \(\wedge_w(A \cup B) = X\). Thus \(\wedge_w(A \cup B) \neq \wedge_w(A) \cup \wedge_w(B)\).

(3) Let \(X = \{a, b, c\}\). Consider the WS \(w = \{\phi, \{a\}, \{b, c\}\}\) on \(X\). Let \(A = \{b\}\) and \(B = \{c\}\). Then \(\vee_w(A) = \phi\), \(\vee_w(B) = \phi\) and \(\vee_w(A \cup B) = \{b, c\}\). Thus \(\vee_w(A \cup B) \neq \vee_w(A) \cup \vee_w(B)\).
Definition 2. In a WS space \((X, w)\) a subset \(A\) is called a \(\wedge_w\)-set (resp. \(\lor_w\)-set) if \(\wedge_w(A) = A\) (resp. \(\lor_w(A) = A\)). By \(\wedge_w\) (resp. \(\lor_w\)), we denote the family of all \(\wedge_w\)-sets (resp. \(\lor_w\)-sets) of the WS space \((X, w)\).

Remark 2. It follows from Proposition 1 (6) and (9) that in a WS \(w\) if \(A \in w\), then \(A\) is a \(\wedge_w\)-set and if \(X - A \in w\) then \(A\) is a \(\lor_w\)-set. Also it is easy to observe form Definition 1 that, \(X\) is a \(\wedge_w\)-set and \(\phi\) is a \(\lor_w\)-set.

Theorem 1. If \(w\) is a WS on \(X\), then

(1) \(\phi\) and \(X\) are \(\lor_w\)-sets (\(\phi\) and \(X\) are \(\wedge_w\)-sets).

(2) The union of \(\lor_w\)-sets is a \(\lor_w\)-set.

(3) The intersection of \(\wedge_w\)-sets is a \(\wedge_w\)-set.

Proof. (1) This follows from Remark 2.

(2) Let \(\{C_\alpha : \alpha \in \Omega\}\) be a family of \(\lor_w\)-sets in a WS on \(X\). Then by Proposition 1 and Definition 2, \(\lor_w(C_\alpha) \subseteq \lor_w[C_\alpha]\) \(\subseteq \lor_w\). Hence \(\lor_w(C_\alpha) = \lor_w[C_\alpha]\).

(3) Let \(\{C_\alpha : \alpha \in \Omega\}\) be a family of \(\wedge_w\)-sets in a WS on \(X\). Then by Proposition 1 and Definition 2, \(\wedge_w(C_\alpha) \subseteq \wedge_w[C_\alpha]\) \(\subseteq \wedge_w\). Hence \(\wedge_w(C_\alpha) = \wedge_w[C_\alpha]\). 

Definition 3. A WS space \((X, w)\) is said to be \(w\)-\(T_1\) if for any pair of distinct points \(x\) and \(y\) of \(X\), there exist a \(w\)-open set \(U\) of \(X\) containing \(x\) but not \(y\) and a \(w\)-open set \(V\) of \(X\) containing \(y\) but not \(x\).

Theorem 2. For a WS space \((X, w)\), the implications \((2) \Rightarrow (3) \Rightarrow (1)\) hold. If \(w\) is GT, then the following properties are equivalent:

(1) \((X, w)\) is \(w\)-\(T_1\);

(2) For each \(x \in X\), the singleton \(\{x\}\) is \(w\)-closed in \((X, w)\);

(3) For each \(x \in X\), the singleton \(\{x\}\) is a \(\wedge_w\)-set.

Proof. (1) \(\Rightarrow\) (2): Let \(y\) be any point of \(X\) and \(x \in X - \{y\}\). There exists \(V_x \in w\) such that \(x \notin V_x\) and \(y \notin V_x\). Hence we have \(X - \{y\} = \cup_{x \in X - \{y\}} V_x\). Therefore, the singleton \(\{y\}\) is \(w\)-closed in \((X, w)\).

(2) \(\Rightarrow\) (3): Let \(x\) be any point of \(X\) and \(y \in X - \{x\}\). Then \(x \in X - \{y\} \in w\) and \(\wedge_w(\{x\}) \subseteq X - \{y\}\). Therefore, \(y \notin \wedge_w(\{x\})\) and \(\wedge_w(\{x\}) \subseteq \{x\}\). This shows that \(\wedge_w(\{x\}) = \{x\}\). Therefore, the singleton \(\{x\}\) is a \(\wedge_w\)-set.

(3) \(\Rightarrow\) (1): Suppose that the singleton \(\{x\}\) is a \(\wedge_w\)-set for each \(x \in X\). Let \(x\) and \(y\) be any distinct points. Then \(y \notin \wedge_w(\{x\})\) and there exists a \(w\)-open set \(U_x\) such that \(x \in U_x\) and \(y \notin U_x\). Similarly, \(x \notin \wedge_w(\{y\})\) and there exists a \(w\)-open set \(U_y\) such that \(y \in U_y\) and \(x \notin U_y\). This shows that \((X, w)\) is \(w\)-\(T_1\).

Theorem 3. For a WS space \((X, w)\), the implications \((2) \Leftrightarrow (3) \Rightarrow (1)\) hold. If \(w\) is a GT, then the following properties are equivalent:
\( (X, w) \) is \( w\)-\( T_1 \).

(2) Every subset of \( X \) is a \( \land_w \)-set.

(3) Every subset of \( X \) is a \( \lor_w \)-set.

**Proof.** It is obvious that (2) \( \iff \) (3).

(1) \( \implies \) (3): Let \( A \) be any subset of \( X \). Since \( A = \cup \{ \{ x \} : x \in A \} \), by Theorem 2 \( A \) is the union of \( w \)-closed sets, hence \( A \) is a \( \lor_w \)-set (by Remark 2 and Theorem 1).

(2) \( \implies \) (1): Let \( x \in X \). Then by (2), \( \{ x \} \) is a \( \land_w \)-set. Let \( p, q \) be any two distinct points of \( X \). Then \( q \notin \land_w(\{ p \}) = \{ p \} \). So by definition of \( \land_w \)-sets, there exists a \( w \)-open set \( U \) such that \( p \in U \) but \( q \notin U \). Similarly the other case can done. Thus \( (X, w) \) is \( w\)-\( T_1 \).

\[ \blacksquare \]

## 4. Generalized \( \land_w \)-sets and generalized \( \lor_w \)-sets

**Definition 4.** In a WS space \( (X, w) \), a subset \( B \) is called a generalized \( \land_w \)-set (briefly g.\( \land_w \)-set) if \( \land_w(B) \subseteq F \) whenever \( B \subseteq F \) and \( F \) is \( w \)-closed. The complement of a g.\( \land_w \)-set is called a g.\( \lor_w \)-set.

**Proposition 2.** In a WS space \( (X, w) \), the following properties hold:

(1) Every \( \land_w \)-set is a g.\( \land_w \)-set;

(2) Every \( \lor_w \)-set is a g.\( \lor_w \)-set.

**Proof.** (1) This follows from Definitions 2 and 4.

(2) Let \( B \) be a \( \lor_w \)-set subset of \( X \). Then \( B = \lor_w(B) \). By Proposition 1 (7), \( \land_w(X - B) = X - \lor_w(B) = X - B \). Thus by (1) and Definition 4, \( B \) is a g.\( \lor_w \)-set.

\[ \blacksquare \]

**Proposition 3.** Let \( (X, w) \) be a WS space. For each \( x \in X \), the following properties hold:

(1) \( \{ x \} \) is \( w \)-open or \( X - \{ x \} \) is a g.\( \land_w \)-set.

(2) \( \{ x \} \) is \( w \)-open or \( \{ x \} \) is a g.\( \lor_w \)-set.

**Proof.** (1) Suppose \( \{ x \} \) is not a \( w \)-open set. Then the only \( w \)-closed set \( F \) containing \( X - \{ x \} \) is \( X \). Thus \( \land_w(X - \{ x \}) \subseteq F = X \) and thus \( X - \{ x \} \) is a g.\( \land_w \)-set of \( X \).

(2) This follows from (1) and Definition 4.

\[ \blacksquare \]

**Proposition 4.** If \( A \) is a g.\( \land_w \)-set of a WS space \( (X, w) \) and \( A \subseteq B \subseteq \land_w(A) \), then \( B \) is a g.\( \land_w \)-set of \( (X, w) \).

**Proof.** Since \( A \subseteq B \subseteq \land_w(A) \), by Proposition 1 (2), (3) \( \land_w(A) = \land_w(B) \). Let \( F \) be any \( w \)-closed subset of \( X \) such that \( B \subseteq F \). Then, \( \land_w(B) = \land_w(A) \subseteq F \), since \( A \) is a g.\( \land_w \)-set.

\[ \blacksquare \]
Proposition 5. A subset $B$ of a WS space $(X, w)$ is a $g. \lor_w$-set if and only if $U \subseteq \lor_w(B)$ whenever $U \subseteq B$ and $U \subseteq w$.

Proof. Let $U$ be a $w$-open subset of $(X, w)$ such that $U \subseteq B$. Then since $X - U$ is $w$-closed and $X - B \subseteq X - U$, we have $\land_w(X - B) \subseteq X - U$ by Definition 4. Hence by Proposition 1 (7) $X - \lor_w(B) \subseteq X - U$. Thus $U \subseteq \lor_w(B)$. Conversely, let $F$ be a $w$-closed subset of $X$ such that $X - B \subseteq F$. Since $X - F$ is $w$-open and $X - F \subseteq B$, by assumption we have $X - F \subseteq \lor_w(B)$. Then $\land_w(X - B) = X - \lor_w(B) \subseteq F$ by Proposition 1 (7). Thus $X - B$ is a $g. \land_w$-set and hence $B$ is a $g. \lor_w$-set.

Corollary 1. Let $B$ be a $g. \lor_w$-set in a WS space $(X, w)$. Then for every $w$-closed set $F$ such that $\lor_w(B) \cup (X - B) \subseteq F$, $X = F$ holds.

Proof. The assumption $\lor_w(B) \cup (X - B) \subseteq F$ implies that $X - F \subseteq (X - \lor_w(B)) \cap B$. Since $B$ is a $g. \lor_w$-set, then by Proposition 5, we have $X - F \subseteq \lor_w(B)$. On the other hand, $X - F \subseteq \lor_w(B) \cap (X - \lor_w(B)) = \phi$. Therefore, we have $X = F$.

Corollary 2. Let $B$ be a $g. \lor_w$-set in a WS space $(X, w)$. Then $\lor_w(B) \cup (X - B)$ is a $w$-closed set if and only if $B$ is a $\lor_w(B)$-set.

Proof. Suppose that $\lor_w(B) = B$, then $\lor_w(B) \cup (X - B) = X$ is $w$-closed. Conversely, by Corollary 1, $X = (X - B) \cup \lor_w(B)$. Thus $(X - \lor_w(B)) \cap B = \phi$. Hence by Proposition 1 (8), $\lor_w(B) = B$.

Definition 5. Let $w$ be a weak structure (WS) on $X$. Then $A \subseteq X$ is called a $w$-generalized closed set (or simply $wg$-closed set) if $c_w(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $w$-open. The complement of a $wg$-closed set is called a $w$-generalized open (or simply $wg$-open) set.

Theorem 4. Let $(X, w)$ be a WS space such that $H \cap c_w(K)$ is $w$-closed for any $w$-closed set $H$ and any subset $K$ of $X$. Then a subset $A$ of $X$ is $wg$-closed if and only if $c_w(A) - A$ contains no nonempty $w$-closed sets.

Proof. Suppose that $A$ is $wg$-closed. Let $F$ be a $w$-closed subset of $c_w(A) - A$. Since $A \subseteq X - F$ and $A$ is $wg$-closed, $c_w(A) \subseteq X - F$ and so $F \subseteq X - c_w(A)$. Therefore, $F = \phi$. Conversely, suppose the condition holds and $A \subseteq M$ and $M \in w$. If $c_w(A) \not\subseteq M$, then $c_w(A) \cap (X - M)$ is a nonempty $w$-closed subset of $c_w(A) - A$. This contradicts the hypothesis. Therefore, $c_w(A) \subseteq M$ which implies that $A$ is $wg$-closed.

Definition 6. A WS space $(X, w)$ is said to be $w$-$T_\frac{1}{2}$ if every $wg$-closed subset of $X$ is $w$-closed.
Let $A$ be a $w$-open set containing $x$. If $w$ is GT, then the following statements are equivalent:

(1) $(X, w)$ is $w$-$T_{\frac{1}{2}}$

(2) For each $x \in X$ the singleton $\{x\}$ is $w$-closed or $w$-open.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $(X, w)$ is $w$-$T_{\frac{1}{2}}$ and let $x \in X$. If $\{x\}$ is not $w$-closed, then $X - \{x\}$ is not $w$-open, and thus $X$ is the only possible $w$-open set containing $X - \{x\}$. Thus $X - \{x\}$ is $wg$-closed. By assumption, $X - \{x\}$ is $w$-closed, that is $\{x\}$ is $w$-open.

(2) $\Rightarrow$ (1). Suppose that every singleton of $X$ is $w$-open or $w$-closed and let $A$ be a $wg$-closed subset of $X$. Let $x \in c_w(A)$. We discuss the following two cases:

(a) $\{x\}$ is $w$-open. Then $\{x\} \cap A \neq \emptyset$, that is $x \in A$.

(b) $\{x\}$ is $w$-closed. Since $A$ is $wg$-closed, it follows from Theorem 4 that $x \notin c_w(A) - A$ and so $x \in A$.

Thus in both cases, $x \in A$. Therefore, $c_w(A) = A$, that is, $A$ is $w$-closed.

Hence, $(X, w)$ is $w$-$T_{\frac{1}{2}}$. ■

**Theorem 6.** For a WS space $(X, w)$, the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) hold. If $w$ is GT, then the following statements are equivalent:

(1) $(X, w)$ is $w$-$T_{\frac{1}{2}}$.

(2) Every $g.\land_w$-set is a $\land_w$-set.

(3) Every $g.\lor_w$-set is a $\lor_w$-set.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $(X, w)$ is $w$-$T_{\frac{1}{2}}$. If $A$ is a $g.\land_w$-set which is not a $\land_w$-set, then since $A \subseteq \land_w(A)$, there exists $x \in \land_w(A)$ such that $x \notin A$. By Theorem 5, $\{x\}$ is $w$-open or $w$-closed. We discuss two cases:

(a) $\{x\}$ is $w$-open. Then $X - \{x\}$ is a $w$-closed set containing $A$ and $A$ is a $g.\land_w$-set. Hence $\land_w(A) \subseteq X - \{x\}$, that is, $x \notin \land_w(A)$. This is a contradiction.

(b) $\{x\}$ is $w$-closed. Then $X - \{x\}$ is a $w$-open set containing $A$, and $\land_w(A) \subseteq X - \{x\}$. This is contray that $x \in \land_w(A)$. This contradiction proves the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Suppose that every $g.\land_w$-set is a $\land_w$-set and let $x \in X$. We will prove that $\{x\}$ is $w$-open or $w$-closed. If $\{x\}$ is not $w$-open, then $X - \{x\}$ is not $w$-closed, and so the only $w$-closed set containing $X - \{x\}$ is $X$. Thus, $X - \{x\}$ is a $g.\land_w$-set. By assumption, $X - \{x\}$ is a $\land_w$-set. Therefore, $X - \{x\}$ is $w$-open, that is, $\{x\}$ is $w$-closed. Hence by Theorem 5, $(X, w)$ is $w$-$T_{\frac{1}{2}}$.

(2) $\Leftrightarrow$ (3). This is obvious. ■
Theorem 7 ([2]). Let \( \tau(\Lambda_w) \) be the topology generated by \( \Lambda_w \). That is \( \tau(\Lambda_w) = \{ V : V = \bigcup_{B \subseteq \Lambda_w} B \} \) is a topology for \( X \).

Definition 7 ([1]). A WS space \((X, w)\) is said to be \( w-R_0 \) if every \( w \)-open set contains the \( w \)-closure of each of its singletons.

Definition 8 ([2]). A WS space \((X, w)\) is said to be

1. \( w-T_0 \) if for any pair of distinct points of \( X \), there exists a \( w \)-open set containing one of the points but not the other.
2. \( w-T_1 \) if for any pair of distinct points \( x \) and \( y \) of \( X \), there exist a \( w \)-open set \( U \) of \( X \) containing \( x \) but not \( y \) and a \( w \)-open set \( V \) of \( X \) containing \( y \) but not \( x \).

Theorem 8 ([2]). For a WS space \((X, w)\), the following properties are equivalent:

1. \((X, w)\) is \( w-T_1 \);
2. \((X, w)\) is \( w-T_0 \) and \( w-R_0 \);
3. \((X, \tau(\Lambda_w))\) is \( T_0 \) and \( R_0 \);
4. \((X, \tau(\Lambda_w))\) is \( T_1 \).

5. Conclusion

The investigation enables us to obtain a unified theory of notions related to different sets for example \( \land \)-sets, \( \lor \)-sets, semi-\( \land \)-sets, semi-\( \lor \)-sets, pre-\( \land \)-sets, pre-\( \lor \)-sets in topological spaces, \( \land_m \)-sets and \( \lor_m \)-sets in \( m \)-spaces and \( \land_\mu \)-sets and \( \lor_\mu \)-sets in GT spaces.

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References

[1] Al-Omari A., Noiri T., Weak separation axioms \( w-R_0 \) and \( w-R_1 \) in weak structures due to Császár, (submitted).
[2] Al-Omari A., Noiri T., Characterizations of \( w-T_0 \) and \( w-R_0 \) via the topology generated by \( \Lambda_w \), Questions and Answers in General Topology, (accepted).
\[\wedge_w\]-sets and \[\vee_w\]-sets in weak structures . . .


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