ON THE RATES OF CONVERGENCE OF CERTAIN
BIVARIATE LINEAR POSITIVE OPERATORS

Abstract. In this paper, we present a sequence of linear positive
bivariate operators and investigate the approximation properties
of them. Next we study the rates of converge of this approximation
by means modulus of continuity and functions from Lipschitz
class. After we give a Voronovskaya type theorem for $n$ Moreover,
we give an $r$th order generalization of these operators. Finally,
we investigate approximation properties of this generalization and
observe the rates of convergence for them.

Key words: linear positive operator, modulus of continuity, or-
der of approximation, Voronovskaya type theorem, Kantorovich
operators, Beta operators.

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1. Introduction

Let $f \in C([0,1])$. The well known Beta operators are defined as follows:

$$B_n (f;x) := \int_0^1 t^{nx-1} (1-t)^{n(1-x)-1} \frac{B(nx,n(1-x))}{B(n,n)} f(t) \, dt,$$

where $n \in \mathbb{N} = \{1, 2, \ldots\}$, $x \in (0,1)$ and $B$ is the familiar Beta function and
we set $B_n(f,k) := f(k)$, $k = 0,1$. Some approximation properties of Beta
operators were studied in [5], [6], [10] and references there in.

Let $x \in [0,1]$ and $f \in C([0,1])$. The Bernstein-Kantorovich operators
are defined by

$$K_n (f;x) := n \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s) \, ds, \ n \in \mathbb{N}.$$

Also some approximation properties of $K_n$ can be viewed in [1].
Now, we construct the tensor product of $B_m$ and $K_n$, $B_{mn} := B_m^{(1)} \circ K_n^{(2)}$, here "$\circ$" denotes the composition and $B_m^{(1)}$ and $K_n^{(2)}$ are parametric extensions of $B_m$ and $K_n$. Clearly $B_{mn}$, $m, n \in \mathbb{N}$ are linear positive operators.

$$B_{mn}(f; x, y) := n \sum_{k=0}^{n} \binom{n}{k} y^k (1 - y)^{n-k} \int_{0}^{1} \int_{0}^{\theta} f(\theta, s) \frac{\theta^{mx-1} (1 - \theta)^{m(1-x)-1}}{B(mx, m(1-x))} d\theta ds,$$

where $f \in C(I^2)$, i.e., continuous real-valued functions defined on $I^2$, $I^2 := [0, 1] \times [0, 1]$.

In this paper, we study the approximation properties of the operators $B_{mn}$ and obtain the rates of convergence by means of modulus of continuity and functions from Lipschitz class. For second order differentiable functions, Voronovskaya [11] was the first to prove a theorem for Bernstein polynomials known as Voronovskaya Theorem. Later on, it was studied by many authors for some other linear positive operators (e.g., [3], [4], [8]). For $B_{mn}$, we will also prove a Voronovskaya type theorem for an arbitrary continuous function by a function having all continuous partial derivatives up to order two in $B_{mn}$. Moreover, we state an $r$th order generalization of $B_{mn}$. It is known that $r$th order generalization of linear positive operators of functions with one variable were introduced in [7], [9]. Finally, we study the approximation properties of this generalization and establish the rates of convergence.

2. Approximation Properties of $B_{mn}$

In this section, we give some approximation properties of $B_{mn}$ on $I^2$.

**Lemma 1.** For all $m, n \in \mathbb{N}$, we have

$$B_{mn}(1; x, y) = 1,$$
$$B_{mn}(\theta; x, y) = x,$$
$$B_{mn}(s; x, y) = y + \frac{1}{2n},$$
$$B_{mn}(\theta^2 + s^2; x, y) = \frac{mx^2 + x}{m + 1} + y^2 + \frac{y(1 - y)}{n} + \frac{y}{n} + \frac{1}{3n^2}.$$

Using Lemma 1 we have the following theorem for the convergence of the operators $B_{mn}$.

**Theorem 1.** Let $f(x, y) \in C(I^2)$, $(x, y) \in I^2$, then $B_{mn}(f; x, y)$ converges to $f(x, y)$ uniformly on $I^2$ i.e.,

$$\lim_{m,n\to\infty} \|B_{mn}(f; x, y) - f(x, y)\|_{C(I^2)} = 0.$$
Proof is clear from Lemma 1 and the well known Volkov’s theorem [10]. Now, we give the following Lemmas which we shall use.

**Lemma 2.** For all $m, n \in N$, we have

\[
B_{mn} (\theta s; x, y) = x \left( y + \frac{1}{2n} \right),
\]

\[
B_{mn} (\theta^2; x, y) = \frac{mx^2 + x}{m + 1},
\]

\[
B_{mn} (\theta^3; x, y) = \frac{(mx + 2) (mx + 1) mx}{(m + 2) (m + 1) m},
\]

\[
B_{mn} (\theta^4; x, y) = \frac{(mx + 3) (mx + 2) (mx + 1) mx}{(m + 3) (m + 2) (m + 1) m},
\]

\[
B_{mn} (s^2; x, y) = y^2 + y \left( 1 - y \right) + \frac{y}{n} + \frac{1}{3n^2},
\]

\[
B_{mn} (s^3; x, y) = \frac{4n (n - 2) (n - 1) y^3 + 18n (n - 1) y^2 + 12ny + 1}{4n^3},
\]

\[
B_{mn} (s^4; x, y) = \frac{(n - 3) (n - 2) (n - 1) y^4 + 8 (n - 2) (n - 1) y^3}{n^3} + \frac{15 (n - 1) y^2 + 5y}{5n^4} + \frac{1}{5n^4}.
\]

**Lemma 3.** For operators $B_{mn}$ we have

\[
B_{mn} \left( (s - y)^2; x, y \right) = \frac{y (1 - y)}{n} + \frac{1}{3n^2},
\]

\[
B_{mn} \left( (\theta - x)^2; x, y \right) = \frac{x (1 - x)}{m + 1},
\]

\[
B_{mn} \left( (\theta - x) (s - y); x, y \right) = 0,
\]

\[
B_{mn} \left( (\theta - x)^4; x, y \right) = \frac{3 (m - 6) x^4 + 6 (6 - m) x^3 + 3 (m - 8) x^2 + 6x}{(m + 3) (m + 2) (m + 1)},
\]

\[
B_{mn} \left( (s - y)^4; x, y \right) = \frac{3 (n - 2) y^4 + (16 - 6n) y^3 + 5 (n - 3) y^2 + 4y}{n^3} + \frac{1}{5n^4}.
\]

In the sequel, we take the operators $B_{mn}$, as follows:

\[
(1) \quad B_{mn} (f; x, y) = n \sum_{k=0}^{n} P_n (y) \int_{0}^{\frac{k+1}{n}} \int_{\frac{k}{n}}^{1} \psi_{m,x} (\theta) f (\theta, s) d\theta ds,
\]

where

\[
(2) \quad P_n (y) = \binom{n}{k} y^k (1 - y)^{n-k}
\]
and
\begin{equation}
\psi_{m,x}(\theta) = \frac{\varrho^{m-1}(1-\theta)^{m(1-x)-1} B(mx, m(1-x)))}{B(mx, m(1-x))} \tag{3}
\end{equation}
for the sake of shortness.

3. Rates of convergence

In this section, we study the rates of convergence in Theorem 1 by means of total modulus of continuity and elements of Lipschitz class.

Let \( f \in C(I^2) \). The total modulus of continuity of \( f \), denoted by \( w(f; \delta) \), is defined by
\[
w(f; \delta) = \max \left\{ \frac{|f(x_1, y_1) - f(x_2, y_2)|}{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}} \leq \delta \right\}.
\]
Moreover, partial modulus of continuity with respect to \( x \) and \( y \) are given by
\[
w^{(1)}(f; \delta) = \max_{0 \leq y \leq 1} \max_{|x_1-x_2| \leq \delta} |f(x_1, y_1) - f(x_2, y)|,
\]
and
\[
w^{(2)}(f; \delta) = \max_{0 \leq x \leq 1} \max_{|y_1-y_2| \leq \delta} |f(x_1, y_1) - f(x, y_2)|,
\]
respectively. It is known that a necessary and sufficient condition for a function \( f \) to be in \( C(I^2) \) is
\[
\lim_{\delta \to 0} w(f; \delta) = 0.
\]
We shall use the following property of the total modulus of continuity:
\[
w(f; \lambda \delta) \leq (1 + [\lambda]) w(f; \delta)
\]
for any \( \lambda \in \mathbb{R} \), here \( [\lambda] \) is the greatest integer that does not exceed \( \lambda \) (the same properties also hold for partial modulus of continuity), and
\[
|f(\theta, s) - f(x, y)| \leq w(f; \delta) \left( 1 + \frac{\sqrt{(\theta - x)^2 + (s - y)^2}}{\delta} \right) \tag{4}
\]
The next result gives the rates of convergence of the sequence \( \{B_{mn}(f; x, y)\} \), \( f \in C(I^2) \), in Theorem 1 by means of the total modulus of continuity.

**Theorem 2.** For all \( f \in C(I^2) \), we have
\[
\|B_{mn}(f; x, y) - f(x, y)\|_{C(I^2)} \leq \frac{3}{2} w(f; \delta_{mn}), \tag{5}
\]
where \( \delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}} \).
Proof. Let $f \in C(I^2)$. Since $B_{mn}$ are linear and monotone from (4) then we get that

$$|B_{mn}(f; x, y) - f(x, y)| \leq B_{mn}(|f(\theta, s) - f(x, y)|; x, y)$$

$$\leq w(f; \delta_{mn}) B_{mn} \left( 1 + \frac{(\theta - x)^2 + (s - y)^2}{\delta_{mn}}; x, y \right)$$

$$= w(f; \delta_{mn}) \left\{ 1 + \frac{n}{\delta_{mn}} \sum_{k=0}^{n} P_n(y) \right\} \times \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{0}^{1} \psi_{m,x}(\theta) \left( \frac{\sqrt{(\theta - x)^2 + (s - y)^2}}{\delta_{mn}} \right) d\theta ds.$$ 

Applying Hölder’s inequality to the inner integral in (6), then (6) turns into the following from,

$$B_{mn} |(f; x, y) - f(x, y)| \leq w(f; \delta_{mn}) \left\{ 1 + \frac{n}{\delta_{mn}} \sum_{k=0}^{n} P_n(y) \right\} \times \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{0}^{1} \left[ (\theta - x)^2 + (s - y)^2 \right] \psi_{m,x}(\theta) d\theta ds \right\} \frac{1}{2} \left\{ \int_{0}^{1} \psi_{m,x}(\theta) d\theta \right\} \frac{1}{2} ds \right\}.$$ 

Again applying Hölder’s inequality to the second (outer) integral, then the last inequality takes the from

$$|B_{mn}(f; x, y) - f(x, y)| \leq w(f; \delta_{mn}) \left\{ 1 + \frac{n}{\delta_{mn}} \sum_{k=0}^{n} P_n(y) \right\} \times \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{0}^{1} \left[ (\theta - x)^2 + (s - y)^2 \right] \psi_{m,x}(\theta) d\theta ds \right\} \frac{1}{2} \left\{ \int_{0}^{1} ds \right\} \frac{1}{2}.$$ 

Finally, applying the Cauchy-Schwarz-Bunyakowsky inequality to the summation in the last inequality, then we obtain the following:

$$|B_{mn}(f; x, y) - f(x, y)| \leq w(f; \delta_{mn}) \left\{ 1 + \frac{n}{\delta_{mn}} \right\} \times \left[ \frac{1}{n} \sum_{k=0}^{n} P_n(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{0}^{1} \left[ (\theta - x)^2 + (s - y)^2 \right] \psi_{m,x}(\theta) d\theta ds \right\} \frac{1}{2},$$
here $P_n(y)$ and $\psi_{m,x}(\theta)$ are defined by (2) and (3), respectively. Using Lemma 1 and Lemma 2 and taking maximum over $I^2$, the desired result is obtained in (5).

We have the following theorem for the partial modulus of continuities.

**Theorem 3.** Let $f$ be continuous on $I^2$ and bounded on $R^2$, then we have

$$|B_{mn}(f;x,y) - f(x,y)| \leq \frac{3}{2} \left\{ w^{(1)}(f;\delta_m) + w^{(2)}(f;\delta_n) \right\},$$

where $\delta_m = \frac{1}{\sqrt{m+1}}$, $\delta_n = \sqrt{\frac{3n+4}{3n^2}}$ and $w^{(1)}$, $w^{(2)}$ are the partial modulus of continuity with respect to $x$ and $y$, respectively.

**Proof.** For proof, we can use similarly way in Theorem 2. Let $f(x,y)$ be continuous on $I^2$ and bounded on $R^2$, then we get that following:

$$|B_{mn}(f;x,y) - f(x,y)| \leq B_{mn}(|f(\theta,s) - f(x,y)|;x,y)$$

$$= B_{mn}(|f(\theta,s) - f(\theta,y) + f(\theta,y) - f(x,y)|;x,y)$$

$$\leq B_{mn}(|f(\theta,s) - f(\theta,y)|;x,y) + B_{mn}(|f(\theta,y) - f(x,y)|;x,y)$$

$$\leq w^{(2)}(f;\delta_n) B_{mn}\left(1 + \frac{|s-y|}{\delta_n};x,y\right)$$

$$+ w^{(1)}(f;\delta_m) B_{mn}\left(1 + \frac{\theta-x}{\delta_m};x,y\right)$$

$$= \frac{3}{2} \left\{ w^{(1)}(f;\delta_m) + w^{(2)}(f;\delta_n) \right\}.$$

Thus, the proof of Theorem 3 is finished.

Now, we will investigate the rates of convergence of $B_{mn}$ by means of the Lipschitz class $\text{Lip}_M(\gamma)$ for $0 < \gamma \leq 1$. Recapulate that $\text{Lip}_M(\gamma)$ is given by

$$|f(x_1,y_1) - f(x_2,y_2)| \leq M \left[ (x_1-x_2)^2 + (y_1-y_2)^2 \right]^\frac{\gamma}{2},$$

here $(x_1,y_1), (x_2,y_2) \in I^2$ and $M > 0$.

**Theorem 4.** Let $f \in \text{Lip}_M(\gamma), 0 < \gamma \leq 1$, then we have

$$\|B_{mn}(f;x,y) - f(x,y)\|_{C(I^2)} \leq \frac{M}{2^\gamma} \delta_{mn}^\gamma,$$

where $\delta_{mn}$ is given by $\delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}} = \sqrt{\delta_m^2 + \delta_n^2}$, which is the same in Theorem 2, and $\delta_m, \delta_n$ are the same as in Theorem 3.
Proof. Let $f \in Lip_M(\gamma)$, $0 < \gamma \leq 1$. Using linearity and monotonicity of $B_{mn}$, we get that

\[ |B_{mn}(f; x, y) - f(x, y)| \leq B_{mn}(|f(\theta, s) - f(x, y)|; x, y) \]

\[ \leq Mn \sum_{k=0}^{n} P_n(y) \int_{k/n}^{1/n} \int_{0}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \]

by (7). Applying Hölder’s inequality to the inner integral in (9), we obtain that

\[ |B_{mn}(f; x, y) - f(x, y)| \leq Mn \sum_{k=0}^{n} P_n(y) \left\{ \int_{0}^{1} \int_{k/n}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \right\}^{2/\gamma} \]

\[ \times \left\{ \int_{0}^{1} \int_{k/n}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \right\}^{2 - \gamma/2} \]

Again, to the outer integral is applied by Hölder’s inequality, then the last inequality takes the following form:

\[ |B_{mn}(f; x, y) - f(x, y)| \]

\[ \leq Mn \sum_{k=0}^{n} P_n(y) \left\{ \int_{0}^{1} \int_{k/n}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \right\}^{2/\gamma} \]

\[ \times \left\{ \int_{0}^{1} \int_{k/n}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \right\}^{2 - \gamma/2} \]

Now, applying Hölder’s inequality for the sum in the last inequality we reach to the following:

\[ |B_{mn}(f; x, y) - f(x, y)| \leq M n^{2 - \gamma/2} \]

\[ \times \left\{ \int_{0}^{1} \int_{k/n}^{1/k} \psi_{m,x}(\theta) \left[ (\theta - x)^2 + (s - y)^2 \right]^{\gamma/2} d\theta ds \right\}^{2 - \gamma/2} . \]
Using Lemma 1, Lemma 2 and taking maximum over $I^2$, (8) is obtained easily, which completes the proof.

4. A Voronovskaya type theorem

In this section, we will give a Voronovskaya type Theorem in $C^2(I^2)$ for operators $B_{mn}$ for $m = n$.

Let $C^2$ denote the space of all functions $f$ having all continuous partial derivatives up to order 2 exist, i.e.:

$$C^2 := \{ f \in C : f_{x^i y^j} \text{ exist for } 0 \leq i, j \leq 2 \text{ and } 0 \leq i + j \leq 2 \text{ is continuous} \},$$

where $f_{x^i y^j} := \frac{\partial^2 f(x,y)}{\partial x^i \partial y^j}$.

Firstly, we need the following lemma.

**Lemma 4.** Let $(x,y) \in I^2$. Then, we get

$$\lim_{m \to \infty} m^2 B_{mn} \left( (\theta - x)^4 ; x, y \right) = 3x^4 - 6x^3 + 3x^2,$$

$$\lim_{n \to \infty} n^2 B_{mn} \left( (s - y)^4 ; x, y \right) = 3y^4 - 6y^3 + 5y^2.$$

**Proof.** By Lemma 3 and the linearity of $B_{mn}$, we may write that

$$m^2 B_{mn} \left( (\theta - x)^4 ; x, y \right)$$

$$= m^2 \left[ \frac{3 (m - 6) x^4 + 6 (6 - m) x^3 + 3 (m - 8) x^2 + 6x}{(m + 3) (m + 2) (m + 1)} \right]$$

$$n^2 B_{mn} \left( (s - y)^4 ; x, y \right)$$

$$= n^2 \left[ \frac{3 (n - 2) y^4 + (16 - 6n) y^3 + 5 (n - 3) y^2 + 4y}{n^3} + \frac{1}{5 n^4} \right].$$

Taking limit as $m \to \infty, n \to \infty$ respectively, then, the proof is complete.

**Theorem 5.** For every $f \in C^2(I^2)$ and $(x,y) \in I^2$, we have

$$\lim_{n \to \infty} n \{ B_{nn} (f; x, y) - f (x, y) \}$$

$$= \frac{1}{2} \{ f_y + x(1-x)f_{xx} + y(1-y)f_{yy} \}.$$
Proof. Let \((x, y) \in I^2\) and \(f_x, f_y, f_{xx}, f_{xy}, f_{yy} \in C^2(I)\). Define the function \(\Phi\) as follows:

\[
\Phi_{(x,y)}(\theta, s) = \begin{cases} 
\frac{f(\theta, s)-f(x,y)-(\theta-x)f_x-(s-y)f_y}{\sqrt{(\theta-x)^2+(s-y)^2}} - \frac{1}{2}\left\{(\theta-x)^2f_{xx}+2(\theta-x)(s-y)f_{xy}+(s-y)^2f_{yy}\right\}, & (\theta, s) \neq (x, y) \\
0, & (\theta, s) = (x, y).
\end{cases}
\]

Then, by assumption we have \(\Phi_{(x,y)}(x, y) = 0\) and the function \(\Phi_{(x,y)}(\cdot, \cdot) \in C^2(I^2)\). Hence, by the Taylor formula for \(f \in C^2(I^2)\), we get

\[
(f(x, y) + (\theta - x)f_x + (s - y)f_y) + \frac{1}{2} \left\{ (\theta - x)^2 f_{xx} + 2(\theta - x)(s - y)f_{xy} + (s - y)^2 f_{yy} \right\} + \Phi_{(x,y)}(\theta, s) \sqrt{(\theta - x)^4 + (s - y)^4}.
\]

We apply the linear operators \(nB_{nn}\) to (13) and using Lemma 2, we have

\[
\begin{align*}
(14) & \quad n \{ B_{nn}(f;x,y) - f(x,y) \} \\
& \quad = \frac{1}{2} \left\{ f_y + \frac{nx(1-x)}{n+1} f_{xx} + \left[ y(1-y) + \frac{1}{3n} \right] f_{yy} \right\} \\
& \quad + nB_{nn}\left( \Phi_{(x,y)}(\theta, s) \sqrt{(\theta - x)^4 + (s - y)^4}; x, y \right).
\end{align*}
\]

If we apply the Cauchy-Schwarz inequality for the second term on the right-hand side of (14) then we conclude that

\[
\begin{align*}
(15) & \quad n \left| B_{nn}\left( \Phi_{(x,y)}(\theta, s) \sqrt{(\theta - x)^4 + (s - y)^4}; x, y \right) \right| \\
& \quad \leq \left[ B_{nn}\left( \Phi^2_{(x,y)}(\theta, s); x, y \right) \right]^{\frac{1}{2}} \\
& \quad \times \left[ \left( (\theta - x)^4 + (s - y)^4 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\
& \quad = \left[ B_{nn}\left( \Phi^2_{(x,y)}(\theta, s); x, y \right) \right]^{\frac{1}{2}} \\
& \quad \times \left[ n^2 B_{nn}\left( (\theta - x)^4; x, y \right) + n^2 B_{nn}\left( (s - y)^4; x, y \right) \right]^{\frac{1}{2}}.
\end{align*}
\]

Let \(\varphi_{(x,y)}(\theta, s) = \Phi^2_{(x,y)}(\theta, s)\). In this case, observe that \(\varphi_{(x,y)}(x, y) = 0\) and \(\varphi_{(x,y)}(\cdot, \cdot) \in C^2(I^2)\). From Theorem 1,

\[
\lim_{n \to \infty} B_{nn}\left( \Phi^2_{(x,y)}(\theta, s); x, y \right) = \lim_{n \to \infty} B_{nn}\left( \varphi_{(x,y)}(\theta, s); x, y \right) = \varphi_{(x,y)}(x, y) = 0.
\]
Using (16) and Lemma 4, we have from (15)

\begin{equation}
\lim_{n \to \infty} nB_{mn} \left( \Phi_{(x,y)} (\theta, s) \sqrt{(\theta - x)^4 + (s - y)^4}; x, y \right) = 0.
\end{equation}

On the other hand, observe that

\begin{equation}
\lim_{n \to \infty} \frac{nx (1 - x)}{n + 1} = x (1 - x), \quad \lim_{n \to \infty} \frac{1}{3n} = 0.
\end{equation}

Then, taking limit as \( n \to \infty \) in (14) and using (17) and (18) we have (12). So the proof is completed.

\section{A Generalization of order \( r \) of \( B_{mn} \)}

Let \( C^r (I^2) \), \( r \in \mathbb{N} \cup \{0\} \), denote the space of all functions \( f \) having all continuous partial derivatives up to order \( r \) exist at \((x, y) \in I^2\). Let \( B_{mn}^r \) denote the following generalization of \( B_{mn} \).

\begin{equation}
B_{mn}^r (f; x, y) = n \sum_{k=0}^{n} P_n (y) \int_{0}^{x} \int_{0}^{s} \psi_{m,x}(\theta) P_{r, (\theta,s)} (x - \theta, y - s) d\theta ds,
\end{equation}

where

\begin{equation}
P_{r, (\theta,s)} (x - \theta, y - s) = \sum_{h=0}^{r} \sum_{i+j=h} \frac{1}{h!} \binom{h}{j} f_{x^i y^j} (\theta, s) (x - \theta)^i (y - s)^j,
\end{equation}

and by the subscripts appeared on \( f_{x^i y^j} \), we denote to write the partial derivatives of \( f \), i.e.: \( f_{x^i y^j} := \frac{\partial^r f(x,y)}{\partial x^i \partial y^j}; r = i + j \). Now let us write

\begin{equation}
(x - \theta, y - s) = u(\alpha, \beta),
\end{equation}

where \((\alpha, \beta)\) is a unit vector, \( u > 0 \) and let us write

\begin{equation}
F(u) = f(\theta + u\alpha, s + u\beta) = f(\theta + (x - \theta), s + (y - s)).
\end{equation}

Clearly Taylor’s formula for \( F(u) \) at \( u = 0 \) turns into Taylor’s formula for \( f(x,y) \) at \((\theta,s)\). Moreover, \( r-th \) derivative takes the form (see [2])

\begin{equation}
F^{[r]}(u) = \sum_{i+j=r} \binom{r}{j} f_{x^i y^j} (\theta + u\alpha, s + u\beta) \alpha^i \beta^j, \quad r \in \mathbb{N}.
\end{equation}
Theorem 6. Let \( f \in C^r (I^2) \) and \( F^{[r]} (u) \in \text{Lip}_M (\gamma) \), then the following inequality
\[
\| B^{[r]}_{mn} (f; x, y) - f (x, y) \|_{C(I^2)} \leq \frac{\gamma MB (\gamma, r)}{(\gamma + r) (r - 1)!} \| B^{[r]}_{mn} (| (x, y) - (\theta, s) |^{r+\gamma}; x, y) \|_{C(I^2)}
\]
holds, where \( F^{[r]} (u) \) is given by (23), \( B (\gamma, r) \) is the familiar Beta function, \( 0 < \gamma \leq 1 \) and \( M > 0 \).

Proof. From (19) and (20), we have
\[
f (x, y) - B^{[r]}_{mn} (f; x, y) = n \sum_{k=0}^{n} P_n (y) \int_{0}^{1} \int_{0}^{\frac{k}{n}} \psi_{m,x} (\theta) \{ f (x, y) - P_{r, (\theta, s)} (x - \theta, y - s) \} \, d\theta \, ds.
\]
Using the integral form of the remainder term appeared in the previous formula, we obtain that
\[
f (x, y) - P_{r, (\theta, s)} (x - \theta, y - s) = \frac{1}{(r - 1)!} \int_{0}^{1} \sum_{i+j=h}^{1} \frac{1}{h!} \binom{h}{j} (x - \theta)^i (y - s)^j \\
\times f_x^i y^j (\theta + t (x - \theta), s + t (y - s)) (1 - t)^{r-1} \, dt.
\]
(26) turns into the following form:
\[
F (u) - \sum_{h=0}^{r} \frac{1}{h!} F^{(h)} (0) \, u^h = \frac{u^r}{(r - 1)!} \int_{0}^{1} \left[ F^{(r)} (tu) - F^{(r)} (0) \right] (1 - t)^{r-1} \, dt.
\]
by using (21)-(23).

Taking (21), (26) and (27) into account and considering the fact that \( F^{[r]} (u) \in \text{Lip}_M (\gamma) \), then it follows that
\[
| f (x, y) - P_{r, (\theta, s)} (x - \theta, y - s) | = \left| F (u) - \sum_{h=0}^{r} \frac{1}{h!} F^{(h)} (0) \, u^h \right|
\]
\[
\begin{align*}
\leq & \frac{|u|^r}{(r-1)!} \int_0^1 \left[ F^{(r)}(tu) - F^{(r)}(0) \right] (1-t)^{r-1} dt \\
\leq & \frac{|u|^{r+\gamma}}{(r-1)!} MB(\gamma + 1, r) \\
\leq & \frac{M}{(r-1)!} \frac{\gamma}{\gamma + r} B(\gamma, r) |u|^{r+\gamma} \\
\leq & \frac{M}{(r-1)!} \frac{\gamma}{\gamma + r} B(\gamma, r) |x - \theta, y - s|^{r+\gamma}.
\end{align*}
\]

Hence combining (25) and (28) we obtain (24), which completes the proof. ■

Now, define the function \( g \in C(I^2) \) as

\[
g(\theta, s) = |(x, y) - (\theta, s)|^{r+\gamma}.
\]

It is clear that \( g(x, y) = 0 \). From Theorem 1 we get that

\[
\|B_{mn}(g; x, y)\|_{C(I^2)} \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty.
\]

From (24), we arrive at the following approximation:

\[
\|B_{mn}^{[r]}(f; x, y) - f(x, y)\|_{C(I^2)} \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty.
\]

Taking Theorem 2 and Theorem 4 into consideration the following results can be obtained from Theorem 6.

**Corollary 1.** Under the conditions of Theorem 6, it follows that

\[
\|B_{mn}^{[r]}(f; x, y) - f(x, y)\|_{C(I^2)} \leq \frac{M}{(r-1)!} \frac{\gamma}{(\gamma + r)} B(\gamma, r) \frac{3}{2} w(g; \delta_{mn}),
\]

here \( \delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}} \) which is the same as in Theorem 2.

**Corollary 2.** Under the conditions of Theorem 6 and assuming that \( g(x, y) \in Lip(\sqrt{2})^r(\gamma) \) in Theorem 4, it follows that

\[
\|B_{mn}^{[r]}(f; x, y) - f(x, y)\|_{C(I^2)} \leq \frac{M}{(r-1)!} \frac{\gamma}{(\gamma + r)} B(\gamma, r) \frac{(\sqrt{2})^r}{2^\gamma} \delta_{mn}^\gamma,
\]

where \( \delta_{mn} = \sqrt{\frac{1}{m+1} + \frac{3n+4}{3n^2}} \) which is the same as in Theorem 2.
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