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ON ALMOST \((\gamma,\gamma')-(\beta,\beta')\)-s-CONTINUOUS FUNCTIONS

Abstract. The aim of this paper is to introduce and study a new class of functions called almost \((\gamma,\gamma')-(\beta,\beta')\)-s-continuous functions in topological spaces by using \((\gamma,\gamma')\)-semiopen sets.

Key words: topological spaces, \((\gamma,\gamma')\)-open set, \((\gamma,\gamma')\)-semiopen set.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Kasahara [3] defined the concept of an operation on topological spaces. Umehara et. al. [5] introduced the notion of \(\tau_{(\gamma,\gamma')}\) which is the collection of all \((\gamma,\gamma')\)-open sets in a topological space \((X,\tau)\). In [1] the authors, introduced the notion of \((\gamma,\gamma')\)-semiopeness and investigated its fundamental properties. The aim of this paper is to introduce and study a new class of functions called almost \((\gamma,\gamma')-(\beta,\beta')\)-s-continuous functions in topological spaces by using \((\gamma,\gamma')\)-semiopen sets.

2. Preliminaries

Definition 1 ([3]). Let \((X,\tau)\) be a topological space. An operation \(\gamma\) on the topology \(\tau\) is a function from \(\tau\) in to power set \(\mathcal{P}(X)\) of \(X\) such that \(V \subset V^\gamma\) for each \(V \in \tau\), where \(V^\gamma\) denotes the value of \(\gamma\) at \(V\). It is denoted by \(\gamma : \tau \to \mathcal{P}(X)\).

Definition 2 ([5]). A subset \(A\) of a topological space \((X,\tau)\) is said to be a \((\gamma,\gamma')\)-open set if for each \(x \in A\) there exist open neighborhoods \(U\) and \(V\) of \(x\) such that \(U^\gamma \cup V^\gamma \subset A\). The complement of a \((\gamma,\gamma')\)-open set is called a \((\gamma,\gamma')\)-closed set. Also \(\tau_{(\gamma,\gamma')}\) denotes set of all \((\gamma,\gamma')\)-open sets in \((X,\tau)\).
Definition 3 ([5]). Let $A$ be a subset of a topological space $(X, \tau)$. A point $x \in A$ is said to be the $(\gamma, \gamma')$-interior point of $A$ if there exist open neighborhoods $U$ and $V$ of $x$ such that $U^\gamma \cup V^{\gamma'} \subset A$ and we denote the set of all such points by $\text{Int}_{(\gamma, \gamma')}(A)$. Thus $\text{Int}_{(\gamma, \gamma')}(A) = \{x \in A : x \in U \in \tau, V \in \tau \text{ and } U^\gamma \cup V^{\gamma'} \subset A\}$. Note that $A$ is $(\gamma, \gamma')$-open if and only if $A = \text{Int}_{(\gamma, \gamma')}(A)$.

A subset $A \subset X$ is called $(\gamma, \gamma')$-closed if and only if $X \setminus A$ is $(\gamma, \gamma')$-open.

Definition 4 ([5]). A point $x \in X$ is called the $(\gamma, \gamma')$-closure point of $A \subset X$, if $(U^\gamma \cup V^{\gamma'}) \cap A \neq \emptyset$ for any open neighborhoods $U$ and $V$ of $x$. The set of all $(\gamma, \gamma')$-closure points of $A$ is called $(\gamma, \gamma')$-closure of $A$ and is denoted by $\text{Cl}_{(\gamma, \gamma')}(A)$. A subset $A$ of $X$ is called $(\gamma, \gamma')$-closed if $\text{Cl}_{(\gamma, \gamma')}(A) \subset A$.

Definition 5 ([1]). A subset $A$ of a topological space $(X, \tau)$ is said to be $(\gamma, \gamma')$-semiopen if there exists a $(\gamma, \gamma')$-open set $O$ such that $O \subset A \subset \text{Cl}_{(\gamma, \gamma')}(O)$. The set of all $(\gamma, \gamma')$-semiopen sets is denoted by $\text{SO}_{(\gamma, \gamma')}(X)$. $A$ is $(\gamma, \gamma')$-semiclosed if and only if $X \setminus A$ is $(\gamma, \gamma')$-semiopen in $X$.

Definition 6 ([1]). Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma, \gamma'$ operators on $\tau$.

1. The intersection of all $(\gamma, \gamma')$-semiclosed sets containing $A$ is called the $(\gamma, \gamma')$-semiclosure of $A$ and is denoted by $s\text{Cl}_{(\gamma, \gamma')}(A)$.

2. The union of all $(\gamma, \gamma')$-semiopen subsets of $A$ is called $(\gamma, \gamma')$-semiinterior of $A$ and is denoted by $s\text{Int}_{(\gamma, \gamma')}(A)$.

Definition 7 ([1]). A point $x \in X$ is said to be $(\gamma, \gamma')$-semi-$\theta$-adherent point of a subset $A$ of $X$ if $s\text{Cl}_{(\gamma, \gamma')}(U) \cap A \neq \emptyset$ for every $U \in \text{SO}_{(\gamma, \gamma')}(X)$. The set of all $(\gamma, \gamma')$-semi-$\theta$-adherent points of $A$ is called the $(\gamma, \gamma')$-semi-$\theta$-closure of $A$ and is denoted by $s_{(\gamma, \gamma')} \text{Cl}_\theta(A)$. A subset $A$ is called $(\gamma, \gamma')$-semi-$\theta$-closed if $s_{(\gamma, \gamma')} \text{Cl}_\theta(A) = A$. A subset $A$ is called $(\gamma, \gamma')$-semi-$\theta$-open if and only if $X \setminus A$ is $(\gamma, \gamma')$-semi-$\theta$-closed.

Definition 8 ([1]). A subset $A$ of a topological space $(X, \tau)$ is said to be $(\gamma, \gamma')$-semiregular, if it is both $(\gamma, \gamma')$-semiopen and $(\gamma, \gamma')$-semiclosed. The class of all $(\gamma, \gamma')$-semiregular sets of $X$ is denoted by $\text{SR}_{(\gamma, \gamma')}(A)$.

Definition 9 ([4]). An operation $\gamma$ on $\tau$ is said to be regular if for any open neighborhoods $U, V$ of $x \in X$, there exists an open neighborhood $W$ of $x$ such that $U^\gamma \cap V^\gamma \supset W^\gamma$.

Definition 10 ([4]). An operation $\gamma$ on $\tau$ is said to be open if for every neighborhood $U$ of $x \in X$, there exists a $\gamma$-open set $B$ such that $x \in B$ and $U^\gamma \supset B$.

Definition 11 ([2]). A subset $A$ of a topological space $(X, \tau)$ is said to be $(\gamma, \gamma')$-s-closed if for every cover $\{V_\alpha : \alpha \in I\}$ of $X$ by $(\gamma, \gamma')$-semiopen sets
of $X$, there exists a finite subset $I_0$ of $I$ such that $A \subset \bigcup_{\alpha \in I_0} s\ Cl(\gamma,\gamma')(V_\alpha)$. If $A = X$, the topological space $(X, \tau)$ is called a $(\gamma, \gamma')$-s-closed space.

**Proposition 1 ([2]).** For any space $X$, the following are equivalent:

1. $X$ is $(\gamma, \gamma')$-s-closed.
2. Every cover of $X$ by $(\gamma, \gamma')$-semiregular sets has a finite subcover.

**Definition 12 ([1]).** A function $f : (X, \tau) \to (Y, \tau)$ is said to be $((\gamma, \gamma'), (\beta, \beta'))$-semicontinuous if for any $(\beta, \beta')$-open set $B$ in $Y$, $f^{-1}(B)$ is $(\gamma, \gamma')$-semiregular in $X$.

**Definition 13.** An operation $\gamma : \tau \to P(X)$ is said to be $\gamma$-open, if $V^\gamma$ is $\gamma$-open for each $V \in \tau$.

**Lemma 1 ([1]).** Let $A$ be a subset of a space $X$. Then we have

1. If $A \in SO(\gamma, \gamma')(X)$, then $s\ Cl(\gamma, \gamma')(A) = s(\gamma, \gamma')\ Cl_\theta(A)$.
2. If $A \in SR(\gamma, \gamma')(X)$, then $A$ is $(\gamma, \gamma')$-semi-$\theta$-closed.

**Proof.** (1) Clearly $s\ Cl(\gamma, \gamma')(A) \subset s(\gamma, \gamma')\ Cl_\theta(A)$. Suppose that $x \notin s\ Cl(\gamma, \gamma')(A)$. Then, for some $(\gamma, \gamma')$-semiregular set $U$, $A \cap U = \emptyset$ and hence $A \cap s\ Cl(\gamma, \gamma')(U) = \emptyset$, since $A \in SO(\gamma, \gamma')(X)$. This shows that $x \notin s(\gamma, \gamma')\ Cl_\theta(A)$. Therefore $s\ Cl(\gamma, \gamma')(A) = s(\gamma, \gamma')\ Cl_\theta(A)$.

(2) This follows from (1).  

**Lemma 2 ([1]).** Let $A$ be a subset of a topological space $(X, \tau)$:

1. If $A \in SO(\gamma, \gamma')(X)$, then $s\ Cl(\gamma, \gamma')(A) \in SR(\gamma, \gamma')(X)$.
2. If $A$ is $(\gamma, \gamma')$-open in $X$, then $s\ Cl(\gamma, \gamma')(A) = \text{Int}(\gamma, \gamma')(\text{Cl}(\gamma, \gamma')(X))$.

**Proof.** (1) Since $s\ Cl(\gamma, \gamma')(A)$ is $(\gamma, \gamma')$-semiregular, we show that $s\ Cl(\gamma, \gamma')(A) \in SO(\gamma, \gamma')(X)$. Since $A \in SO(\gamma, \gamma')(X)$, then for $(\gamma, \gamma')$-open set $U$ of $X$, $U \subset A \subset \text{Cl}(\gamma, \gamma')U$. Therefore we have, $U \subset s\ Cl(\gamma, \gamma')(U) \subset s\ Cl(\gamma, \gamma')(A) \subset s\ Cl(\gamma, \gamma')\text{Cl}(\gamma, \gamma')(U)) = \text{Cl}(\gamma, \gamma')(U)$ or $U \subset s\ Cl(\gamma, \gamma')(A) \subset \text{Cl}(\gamma, \gamma')(U)$ and hence $s\ Cl(\gamma, \gamma')(A) \in SO(\gamma, \gamma')(X)$.

**3. Almost $(\gamma, \gamma')$-$(\beta, \beta')$-s-continuous functions**

**Definition 14.** A function $f : (X, \tau) \to (Y, \tau)$ is said to be almost $(\gamma, \gamma')$-$(\beta, \beta')$-s-continuous if for each point $x \in X$ and each $V \in SO(\beta, \beta')(Y)$, there exists a $(\gamma, \gamma')$-open set $U$ containing $x$ such that $f(U) \subseteq s\ Cl(\gamma, \gamma')(V)$.

**Example 1.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Define the operations $\gamma : \tau \to P(X)$, $\gamma' : \tau \to P(X)$, $\beta : \sigma \to P(X)$ and $\beta' : \sigma \to P(X)$ by

\[
A^\gamma = \begin{cases} A & \text{if } b \notin A, \\ Cl(A) & \text{if } b \in A, \end{cases} \quad A'^\gamma = \begin{cases} Cl(A) & \text{if } b \notin A, \\ A & \text{if } b \in A, \end{cases}
\]
\[ A^β = \begin{cases} A & \text{if } a \in A \\ A \cup \{a\} & \text{if } a \notin A \end{cases} \quad \text{and} \quad A'^β = \begin{cases} A & \text{if } c \in A \\ A \cup \{c\} & \text{if } c \notin A \end{cases} \]

Clearly, \( τ_{(γ,γ')} = \{0, X, \{b\}, \{a,b\}, \{a,c\}\} \) and \( SO(β,β')(Y) = \{0, X, \{a,c\}\} \).

Theorem 1. The following statements are equivalent for a function \( f : (X, τ) \to (Y, σ) \):

1. \( f \) is almost \((γ, γ')-(β, β')\)-s-continuous.
2. For each \( x \in X \) and \( V \in SR(β,β')(Y) \), there exists a \((γ, γ')\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).
3. \( f^{-1}(V) \) is \((γ, γ')\)-open as well as \((γ, γ')\)-closed in \( X \) for every \( V \in SR(β,β')(Y) \).
4. \( f^{-1}(V) \subseteq Int(γ,γ')(f^{-1}(sCl(β,β')(V))) \) for every \( V \in SO(β,β')(Y) \).
5. \( Cl(γ,γ')(f^{-1}(sInt(β,β')(V))) \subseteq f^{-1}(V) \) for every \((β, β')\)-seminclosed set \( V \) of \( Y \).
6. \( Cl(γ,γ')(f^{-1}(V)) \subseteq f^{-1}(sCl(β,β')(V)) \) for every \( V \in SO(β,β')(Y) \), where \( γ \) and \( γ' \) are open.

Proof. (1) \( \Rightarrow \) (2): Let \( x \in X \) and \( V \in SR(β,β')(Y) \). There exists a \((γ, γ')\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq sCl(β,β')(V) = V \).

(2) \( \Rightarrow \) (3): Let \( V \in SR(β,β')(Y) \) and \( x \in f^{-1}(V) \). Then \( f(U) \subseteq V \) for some \((γ, γ')\)-open set \( U \) of \( X \) containing \( x \) hence \( x \in U \subseteq f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \((γ, γ')\)-open in \( X \). Since \( Y \setminus V \in SR(β,β')(Y) \), \( f^{-1}(Y \setminus V) \) is also \((γ, γ')\)-open and hence \( f^{-1}(V) \) is \((γ, γ')\)-clopen in \( X \).

(3) \( \Rightarrow \) (4): Let \( V \in SO(β,β')(Y) \). Then by Lemma 2, \( V \subseteq sCl(β,β')(V) \) and \( sCl(β,β')(V) \in SR(β,β')(Y) \). By (3), we have \( f^{-1}(V) \subseteq f^{-1}(sCl(β,β')(V)) \) and \( f^{-1}(sCl(β,β')(V)) \) is \((γ, γ')\)-open in \( X \). Therefore, we obtain \( f^{-1}(V) \subseteq Int(γ,γ')(f^{-1}(sCl(β,β')(V))) \).

(4) \( \Rightarrow \) (5): Let \( V \) be a \((β, β')\)-seminclosed subset of \( Y \). By (4), we have \( f^{-1}(Y \setminus V) \subseteq Int(γ,γ')(f^{-1}(sCl(β,β')(Y \setminus V))) = Int(γ,γ')(f^{-1}(Y \setminus sInt(β,β')(V))) = X \setminus Cl(γ,γ')(f^{-1}(sInt(β,β')(V))) \). Therefore, we obtain \( Cl(γ,γ')(f^{-1}(sInt(β,β')(V))) \subseteq f^{-1}(V) \).

(5) \( \Rightarrow \) (6): Let \( V \in SO(β,β')(Y) \). Then \( sCl(β,β')(V) \in SR(β,β')(Y) \). By Lemma 2 and (5) we obtain \( Cl(γ,γ')(f^{-1}(V)) \subseteq Cl(γ,γ')(f^{-1}(sCl(β,β')(V))) \subseteq f^{-1}(sCl(β,β')(V)) \).

(6) \( \Rightarrow \) (1): Let \( x \in X \) and \( V \in SO(β,β')(Y) \). By Lemma 2, we have \( sCl(γ,γ')(X) \subseteq SR(γ,γ')(X) \) and \( f(x) \notin Y \setminus sCl(β,β')(Y \setminus sCl(β,β')(V)) \). Thus, by (6) we obtain \( x \notin Cl(γ,γ')(f^{-1}(Y \setminus sCl(β,β')(V))) \). There exists a \((γ, γ')\)-open set \( U \) of \( x \) such that \( U \cap f^{-1}(Y \setminus sCl(β,β')(V)) = \emptyset \). Therefore, we have \( f(U) \cap (Y \setminus sCl(β,β')(V)) = \emptyset \) and hence \( f(U) \subseteq sCl(β,β')(V) \). This shows that \( f \) is almost \((γ, γ')-(β, β')\)-s-continuous. \( \blacksquare \)
Theorem 2. The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is almost \((\gamma, \gamma')-(\beta, \beta')\)-s-continuous.
2. For each \( x \in X \) and each \( V \in SR_{(\beta, \beta')}(Y) \), there exists a \((\gamma, \gamma')\)-clopent set \( U \) containing \( x \) such that \( f(U) \subseteq V \).
3. For each \( x \in X \) and each \( V \in SO_{(\beta, \beta')}(Y) \), there exists a \((\gamma, \gamma')\)-open set \( U \) containing \( x \) such that \( f(\text{Cl}_{(\gamma, \gamma)}(U)) \subseteq s\text{Cl}_{(\beta, \beta)}(V) \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( x \in X \) and \( V \in SR_{(\beta, \beta')}(Y) \). By Theorem 1, \( f^{-1}(V) \) is \((\gamma, \gamma')\)-clopent in \( X \). Put \( U = f^{-1}(V) \), then \( x \in U \) and \( f(U) \subseteq V \). The proof of the other implications are obvious.

**Theorem 3.** The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is almost \((\gamma, \gamma')-(\beta, \beta')\)-s-continuous.
2. \( f(\text{Cl}_{(\gamma, \gamma)}(A)) \subseteq s_{(\beta, \beta')} \text{Cl}_\theta(f(A)) \) for every subset \( A \) of \( X \).
3. \( \text{Cl}_{(\gamma, \gamma)}(f^{-1}(B)) \subseteq f^{-1}(s_{(\beta, \beta')} \text{Cl}_\theta(B)) \) for every subset \( B \) of \( Y \).
4. \( f^{-1}(F) \) is \((\gamma, \gamma')\)-closed in \( X \) for every \((\beta, \beta')\)-semi-\( \theta \)-closed set \( F \) of \( Y \).
5. \( f^{-1}(V) \) is \((\gamma, \gamma')\)-open in \( X \) for every \((\beta, \beta')\)-semi-\( \theta \)-open set \( V \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( B \) be any subset of \( Y \) and \( x \notin f^{-1}(s_{(\beta, \beta')} \text{Cl}_\theta(B)) \). Then \( f(x) \notin s_{(\beta, \beta')} \text{Cl}_\theta(B) \) and there exists \( V \in SO_{(\beta, \beta')}(Y, f(x)) \) such that \( s_{(\beta, \beta')} \text{Cl}_\theta(f(A)) \cap B = \emptyset \). By (1), there exists a \((\gamma, \gamma')\)-open set \( U \) containing \( x \) such that \( f(U) \subseteq s_{(\beta, \beta')} \text{Cl}_\theta(f(A)) \). Hence \( f(U) \cap B = \emptyset \) and \( U \cap f^{-1}(B) = \emptyset \). Consequently, we obtain \( x \notin f^{-1}(s_{(\beta, \beta')} \text{Cl}_\theta(f(A))) \).

(2) \( \Rightarrow \) (3): Let \( A \) be any subset of \( X \). By (2), we have \( \text{Cl}_{(\gamma, \gamma)}(A) \subseteq \text{Cl}_{(\gamma, \gamma)}(f^{-1}(f(A))) \subseteq f^{-1}(s_{(\beta, \beta')} \text{Cl}_\theta(f(A))) \) and hence \( f(\text{Cl}_{(\gamma, \gamma)}(A)) \subseteq s_{(\beta, \beta')} \text{Cl}_\theta(f(A)) \).

(3) \( \Rightarrow \) (4): Let \( F \) be any \((\beta, \beta')\)-semi-\( \theta \)-closed set of \( Y \). Then, by (3), we have \( f(\text{Cl}_{(\gamma, \gamma)}(f^{-1}(F))) \subseteq s_{(\beta, \beta')} \text{Cl}_\theta(f(f^{-1}(F))) \subseteq s_{(\beta, \beta')} \text{Cl}_\theta(f) = F \). Therefore, we have \( \text{Cl}_{(\gamma, \gamma)}(f^{-1}(F)) \subseteq f^{-1}(F) \) and hence \( \text{Cl}_{(\gamma, \gamma)}(f^{-1}(F)) = f^{-1}(F) \). This shows that \( f^{-1}(F) \) is \((\gamma, \gamma')\)-closed set in \( X \).

(4) \( \Rightarrow \) (5): This is obvious.

(5) \( \Rightarrow \) (1): Let \( x \in X \) and \( V \in SO_{(\beta, \beta')}(Y, f(x)) \). By Lemmas 1 and 2, \( s_{(\beta, \beta')} \text{Cl}(V) \) is \((\beta, \beta')\)-\( \theta \)-open in \( Y \). Put \( U = f^{-1}(s_{(\beta, \beta')} \text{Cl}(V)) \). Then by (5), \( U \) is \((\gamma, \gamma')\)-open containing \( x \) and \( f(U) \subseteq s_{(\beta, \beta')} \text{Cl}(V) \). Thus, \( f \) is almost \((\gamma, \gamma')-(\beta, \beta')\)-s-continuous.

**Definition 15.** A point \( x \in X \) is said to be a \((\gamma, \gamma')\)-\( \theta \)-adherent point of a subset \( A \) of \( X \) if \( \text{Cl}_{(\gamma, \gamma)}(U) \cap A \neq \emptyset \) for every \((\gamma, \gamma')\)-open set \( U \) containing \( x \).

The set of all \((\gamma, \gamma')\)-\( \theta \)-adherent points of \( A \) is called the \((\gamma, \gamma')\)-\( \theta \)-closure of \( A \) and is denoted by \( \text{Cl}_{(\gamma, \gamma)}(A) \). Note that a subset \( A \) is called \((\gamma, \gamma')\)-\( \theta \)-closed...
if $\text{Cl}_{(\gamma,\gamma')}(A) = A$. The complement of a $(\gamma,\gamma')$-$\theta$-closed set is called a $(\gamma,\gamma')$-$\theta$-open set.

The proof of the following theorem is similar to Theorem 3 and thus omitted.

**Theorem 4.** The following statements are equivalent for a function $f : (X, \tau) \to (Y, \tau)$:

1. $f$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous.
2. $\text{Cl}_{(\gamma,\gamma')}((f^{-1}(A))) \subseteq f^{-1}(s_{(\beta,\beta')}\text{Cl}_{\theta}(A))$ for every subset $A$ of $Y$.
3. $f(\text{Cl}_{(\gamma,\gamma')}(B)) \subseteq s_{(\beta,\beta')}\text{Cl}_{\theta}(f(B))$ for every subset $B$ of $X$.
4. $f^{-1}(F)$ is $(\gamma,\gamma')$-$\theta$-closed in $X$ for every $(\beta,\beta')$-semi-$\theta$-closed set $F$ of $Y$.
5. $f^{-1}(V)$ is $(\gamma,\gamma')$-$\theta$-open in $X$ for every $(\beta,\beta')$-semi-$\theta$-open set $V$ of $Y$.

**Theorem 5.** If $f : (X, \tau) \to (Y, \tau)$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous and $A$ is $(\gamma,\gamma')$-s-closed in $X$, then $f(A)$ is $(\beta,\beta')$-s-closed in $Y$.

**Proof.** Let $\{V_\alpha : \alpha \in I\}$ be any cover of $f(A)$ by $(\beta,\beta')$-semiregular sets of $Y$. By Theorem 1, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of $A$ by $(\gamma,\gamma')$-clopen sets of $X$. By Proposition 1, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \cup s\text{Cl}_{(\gamma,\gamma')}\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ and hence $f(A) \subseteq \cup s\text{Cl}_{(\beta,\beta')}\{V_\alpha : \alpha \in I_0\}$. Hence $f(A)$ is $(\beta,\beta')$-semiclosed relative to $Y$. $\blacksquare$

**Theorem 6.** If $f : (X, \tau) \to (Y, \tau)$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous surjection and $X$ is $(\gamma,\gamma')$-s-closed, then $Y$ is $(\beta,\beta')$-s-closed.

**Proof.** The proof is clear. $\blacksquare$

**Theorem 7.** Let $f : (X, \tau) \to (Y, \tau)$ be a function and $x \in X$. If there exists a $(\gamma,\gamma')$-open set $N$ of $X$ containing $x$ such that the restriction $f_N$ of $f$ to $N$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous at $x$, then $f$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous at $x$, where $\gamma$ and $\gamma'$ are regular.

**Proof.** Let $U$ be any $(\gamma,\gamma')$-semiregular set containing $f(x)$. Since $f_N$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous at $x$, there is a $(\gamma,\gamma')$-open set $V$ containing $x$ such that $x \in N \cap V$ and $f(N \cap V) \subseteq s\text{Cl}_{(\beta,\beta')}(U) = U$ or $f(N \cap V) \subseteq U$. Since $\gamma$ and $\gamma'$ are regular, $N \cap V$ is a $(\gamma,\gamma')$-open set containing $x$. Hence $f$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous at $x$. $\blacksquare$

**Theorem 8.** Let $X_1$, $X_2$ be $(\gamma,\gamma')$-closed sets in a topological space $(X, \tau)$ and $X = X_1 \cup X_2$. If $f : X \to Y$ be a function and $f_{X_1}$ and $f_{X_2}$ are almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous functions, then $f$ is almost $(\gamma,\gamma')$-$(\beta,\beta')$-s-continuous, where $\gamma$ and $\gamma'$ are regular.
Proof. Let $A$ be a $(\beta, \beta')$-semiregular subset of $Y$. Since $f_{X_1}$ and $f_{X_2}$ are both almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous, $(f_{X_1})^{-1}(A)$ and $(f_{X_2})^{-1}(A)$ are both $(\gamma, \gamma')$-clopen subsets of $X$ and $f^{-1}(A) = (f_{X_1})^{-1}(A) \cup (f_{X_1})^{-1}(A)$. Since $\gamma$ and $\gamma'$ are regular, $f^{-1}(A)$ is the union of two $(\gamma, \gamma')$-clopen sets and is thus $(\gamma, \gamma')$-clopen in $X$. Hence $f$ is almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous function.

Remark 1. The following example shows that the regularity on $\gamma$ and $\gamma'$ of Theorem 8 cannot be removed.

Example 2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Define the operations $\gamma : \tau \to \mathcal{P}(X)$, $\gamma' : \tau \to \mathcal{P}(X)$, $\beta : \sigma \to \mathcal{P}(X)$ and $\beta' : \sigma \to \mathcal{P}(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } b \notin A \\ \text{Cl}(A) & \text{if } b \in A, \end{cases} \quad A^\gamma' = \begin{cases} \text{Cl}(A) & \text{if } b \notin A \\ A & \text{if } b \in A, \end{cases}$$

$$A^\beta = \begin{cases} A & \text{if } a \in A \\ A \cup \{a\} & \text{if } a \notin A \end{cases} \quad A^\beta' = \begin{cases} A & \text{if } c \in A \\ A \cup \{c\} & \text{if } c \notin A. \end{cases}$$

Then, it is shown that $\gamma'$ is not regular. Let $X_1 = \{b\}$, $X_2 = \{a, c\}$ and $f : (X, \tau) \to (Y, \tau)$ be an identity function. Clearly, $f_{X_1}$ and $f_{X_2}$ are almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous functions but $f$ is almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous function.

Theorem 9. If $f : X \to Y$ be a function $f_{X_1}$ and $f_{X_2}$ are both almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous at a point $x \in X = X_1 \cup X_2$, then $f$ is almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous at $x$, where $\gamma$ and $\gamma'$ are regular operations.

Proof. Let $U$ be any $(\gamma, \gamma')$-semiregular set containing $f(x)$. Since $x \in X_1 \cap X_2$ and $f_{X_1}$, $f_{X_2}$ are both almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous at a point $x$, there exists $(\gamma, \gamma')$-open sets $V_1$ and $V_2$ of $X$, respectively containing $x$ such that $x \in X_1 \cap V_1, f(X_1 \cap V_1) \subseteq U$ and $x \in X_2 \cap V_2, f(X_2 \cap V_2) \subseteq U$. Since $X = X_1 \cup X_2, f(V_1 \cap V_2) = f(X_1 \cap V_1 \cap V_2) \cup f(X_2 \cap V_1 \cap V_2) \subseteq f(X_1 \cap V_1) \cup f(X_2 \cap V_2) \subseteq U$. Since $\gamma$ and $\gamma'$ are regular, $V_1 \cap V_2 = V$ is a $(\gamma, \gamma')$-open set containing $x$ such that $f(V) \subseteq U$ and hence $f$ is almost $(\gamma, \gamma')-(\beta, \beta')$-s-continuous by Theorem 1.

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