a-LOCAL FUNCTION AND ITS PROPERTIES
IN IDEAL TOPOLOGICAL SPACES

ABSTRACT. In this paper, we introduce the notation of a-local function and study its properties in ideal topological space. We construct a topology \( \tau^a = \beta(I, \tau) \) for \( X \) by using \( a \)-open set and an \( I \) on \( X \). We defined \( a \)-compatible of \( \tau \) with ideal and show that \( \tau \) is \( a \)-compatible with \( I \) then \( \tau^a = \beta(I, \tau) \), where \( \beta(I, \tau) = \{ V-I : V \in \tau^a(x), I \in I \} \) is a basis of \( \tau^a \). Also, The relationships other local functions such as local function [12, 6] and semi-local function [7] are introduced.

KEY WORDS: ideal topological space, \( a \)-local function, \( a \)-open set.

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1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [13]. Jankovic and Hamlett [6] investigated further properties of ideal space. In this paper, we introduce the notation of \( a \)-local function and study its properties in ideal topological space. We construct a topology \( \tau^a = \beta(I, \tau) \) for \( X \) by using \( a \)-open set and an \( I \) on \( X \). We defined \( a \)-compatible of \( \tau \) with ideal and show that \( \tau \) is \( a \)-compatible with \( I \) then \( \tau^a = \beta(I, \tau) \), where \( \beta(I, \tau) = \{ V-I : V \in \tau^a(x), I \in I \} \) is a basis of \( \tau^a \) (Theorem 4). Also, The relationships other local functions such as local function [12, 6] and semi-local function [7] are introduced.

2. Preliminaries

A subset \( A \) of a space \( (X, \tau) \) is said to be regular open (resp. regular closed) [10] if \( A = \text{int}(\text{cl}(A)) \) (resp. \( A = \text{cl}(\text{int}(A)) \)). \( A \) is called \( \delta \)-open [11] if for each \( x \in A \), there exist a regular open set \( G \) such that \( x \in G \subset A \). The complement of \( \delta \)-open set is called \( \delta \)-closed. A point \( x \in X \) is called a
δ-cluster point of A if \( \text{int}(\text{cl}(U)) \cap A \neq \emptyset \) for each open set \( U \) containing \( x \). The set of all δ-cluster points of \( A \) is called the δ-closure of \( A \) and is denoted by \( \text{cl}_\delta(A) \) [11]. The set δ-interior of \( A \) [11] is the union of all regular open sets of \( X \) contained in \( A \) and its denoted by \( \text{int}_\delta(A) \). \( A \) is δ-open if \( \text{int}_\delta(A) = A \). δ-open sets forms a topology \( \tau^\delta \). The collection of all δ-open sets in \( X \) is denoted by \( \delta O(X) \). A subset \( A \) of a space \( (X, \tau) \) is said to be semi-open [9] if \( A \subset \text{cl}(\text{int}(A)) \). The complement of semi-open is said to be semi-closed. The collection of all semi-open sets in \( X \) is denoted by \( \text{SO}(X) \). The semi-closure of \( A \) in \( (X, \tau) \) is defined by the intersection of all semi-closed sets containing \( A \) and is denoted by \( \text{scl}(A) \) [1].

A subset \( A \) of a space \( (X, \tau) \) is said to be a-open (resp. a-closed) \([2, 3]\) if \( A \subset \text{int}(\text{cl}(\text{int}_\delta(A))) \) (resp. \( \text{cl}(\text{int}(\text{cl}_\delta(A))) \subset A \). For a topological space \( (X, \tau) \), the family of all a-open sets of \( X \) forms a topology \([2, 3]\), denoted by \( \tau^a \), for \( X \). The collection of all a-open sets containing \( x \) in \( X \) is denoted by \( \tau^a(x) \). Let \( A \) be a subset of a space \( X \). The intersection of all a-closed sets containing \( A \) is called a-closure of \( A \) \([3]\) and is denoted by \( a\text{cl}(A) \). The a-interior of \( A \), denoted by \( a\text{Int}(A) \), is defined by the union of all a-open sets contained in \( A \) \([3]\).

An ideal \( I \) on a topological space \( (X, \tau) \) is a nonempty collection of subsets of \( X \) which satisfies the following conditions:

1. \( A \in I \) and \( B \subset A \) implies \( B \in I \);
2. \( A \in I \) and \( B \in I \) implies \( A \cup B \in I \).

An ideal topological space is a topological space \( (X, \tau) \) with an ideal \( I \) on \( X \) and if \( P(X) \) is the set of all subsets of \( X \), a set operator \((\cdot)^* : P(X) \to P(X) \), called a local function \([12, 6]\) of \( A \) with respect to \( \tau \) and \( I \) is defined as follows: for \( A \subseteq X \),

\[
A^*(I, \tau) = \{ x \in X \mid U \cap A \notin I, \text{ for every } U \in \tau(x) \}
\]

where \( \tau(x) = \{ U \in \tau \mid x \in U \} \). A Kuratowski closure operator \( \text{Cl}^*(x) = A \cup A^*(I, \tau) \). When there is no chance for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \) and \( \tau^* \) for \( \tau^*(I, \tau) \). \( X^* \) is often a proper subset of \( X \). The hypothesis \( X = X^* \) \([5]\) is equivalent to hypothesis \( \tau \cap I = \emptyset \). For every ideal topological space there exits a topology \( \tau^*(I) \) finer than \( \tau \) generated by \( \beta(I, \tau) = \{ U-A \mid U \in \tau \text{ and } A \in I \} \), but in general \( \beta(I, \tau) \) is not always topology \([6]\). Let \( (X, \tau, I) \) be an ideal topological space and \( A \) be a subset of \( \tau \). Then \( A_\tau(I, \tau) = \{ x \in X : U \cap A \notin I \text{ for every } U \in \text{SO}(X, x) \} \) is called semi local function of \( A \) with respect to \( I \) and \( \tau \) \([7]\). Let \( (X, I, \tau) \) be an ideal topological space. We say that the topology \( \tau \) is compatible with the \( I \), denoted \( \tau \sim I \), if the following hold for every \( A \subset X \), if for every \( x \in A \) there exists a \( U \in \tau \) such that \( U \cap A \in I \), then \( A \in I \) \([6]\).
Lemma 1 ([4]). Let \((X,\tau,\mathcal{I})\) be an ideal topological space, and \(A, B\) subsets of \(X\). Then the following properties hold:

1. If \(A \subseteq B\), then \(A^* \subseteq B^*\);
2. If \(U \in \tau\), then \(U \cap A^* \subset (U \cap A)^*\);
3. \(A^* = cl(A^*) \subset cl(A)\);
4. \((A \cup B)^* = A^* \cup B^*\);
5. \((A \cap B)^* \subset A^* \cup B^*\).

3. \(a\)-local function

Definition 1. Let \((X,\tau,\mathcal{I})\) be an ideal in topological space and \(A\) be a subset of \(X\). Then \(A^{a*}(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}, \text{ for every } U \in \tau^a(x)\}\) is called \(a\)-local function of \(A\) with respect to \(\mathcal{I}\) and \(\tau\). We denote simply \(A^{a*}\) for \(A^{a*}(\mathcal{I},\tau)\).

Remark 1. The notation of the local function, semi local function are independent with \(a\)-local function notation as the following example.

Example 1. Let \(X = \{x, y, w, z\}\) with a topology \(\tau = \{\emptyset, X, \{x, y\}\}\) and \(\mathcal{I} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}\). Take \(A = \{w, z\}\). Then \(A^* = \{\emptyset\}, A_* = \{z\}, A^{a*} = X\).

Remark 2. (1) The minimal ideal is \(\{\emptyset\}\) in any ideal topological space \((X,\tau,\mathcal{I})\) and the maximal ideal is \(P(X)\). It can be deduce that \(A^{a*}(\{\emptyset\}) = aCl(A) \neq cl(A)\) and \(A^{a*}(P(X)) = \emptyset\) for every \(A \subset X\).

(2) If \(A \in \mathcal{I}\), then \(A^{a*} = \emptyset\).

(3) Neither \(A \subset A^{a*}\) nor \(A^{a*} \subset A\) in general.

Theorem 1. Let \((X,\tau,\mathcal{I})\) an ideal in topological space and \(A, B\) subsets of \(X\). Then for \(a\)-local functions the following properties hold:

1. \((\emptyset)^{a*} = \emptyset\),
2. If \(A \subset B\), then \(A^{a*} \subset B^{a*}\),
3. For another ideal \(J \supset I\) on \(X\), \(A^{a*}(J) \subset A^{a*}(\mathcal{I})\),
4. \(A^{a*} \subset aCl(A)\),
5. \(A^{a*}(\mathcal{I}) = aCl(A^{a*}) \subset aCl(A)\) (i.e \(A^{a*}\) is an \(a\)-closed subset of \(aCl(A)\)),
(6) \((A^a)^a \subset A^a\),
(7) \((A \cup B)^a = A^a \cup B^a\),
(8) \(A^a - B^a = (A-B)^a - B^a \subset (A-B)^a\),
(9) If \(U \in \tau^a\), then \(U \cap A^a = U \cap (U \cap A)^a \subset (U \cap A)^a\),
(10) If \(U \in I\), then \((A-U)^a \subset A^a = (A \cup U)^a\).

Proof. (1) This prove is obvious.

(2) Let \(x \in A^a\), then \(U \cap A \notin I\) for every \(U \in \tau^a(x)\). Therefore \(U \cap B \notin I\) for each \(U \in \tau^a(x)\). Since \(A \subset B\) implies that \(U \cap A \subset U \cap B\). If \(U \cap B \in I\) then, \(U \cap A \in I\). Hence \(x \in B^a\) and \(A^a \subset B^a\).

(3) Let \(x \in A^a(J)\). Then for every \(\tau^a(x)\), \(U \cap A \notin J\). This implies that \(U \cap A \notin I\), so \(x \in A^a(I)\). Hence \(A^a(J) \subset A^a(I)\).

(4) Let \(x \in A^a\). Then for every \(a\)-open set containing \(x\), \(U \cap A \notin I\). This implies that \(U \cap A \neq \emptyset\). Hence \(x \in a-cl(A)\).

(5) \(A^a \subset aCl(A^a)\) hold in general. Let \(x \in aCl(A^a)\). Then \(A^a \cap U \neq \emptyset\) for every \(U \in \tau^a(x)\). Therefore, there exist some \(y \in A^a \cap U\) and \(U \in \tau^a(x)\) since \(y \in A^a\), \(A \cap U \notin I\) and hence \(x \in A^a\). Thus \(aCl(A^a) \subset A^a\). Now, let \(aCl(A^a) = A^a\), Then \(A \cap U \notin I\) for every \(U \in \tau^a(x)\). This implies that \(A \cap U \neq \emptyset\) for every \(U \in \tau^a(x)\) and so, \(x \in aCl(X, x)\). Consequently, \(A^a = aCl(A^a) \subset aCl(A)\) and \(A^a\) is an \(a\)-closed.

(6) Let \(x \in (A^a)^a\). Then, for every \(U \in \tau^a(x)\), \(A^a \cap U \notin I\) and hence \(A^a \cap U \neq \emptyset\) for every \(U \in \tau^a(x)\). Thus we have \(A \cap U \notin I\) and \(x \in A^a\).

(7) \(A \subset A \cup B\), and \(B \subset A \cup B\) and \(A^a \cup B^a \subset (A \cup B)^a\) by (1). Conversely, let \(x \in (A \cup B)^a\). Then for every \(U \cap (A \cup B) \notin I = (U \cap A) \cup (U \cap B) \notin I\). Therefore, \((U \cap A) \notin I\) or \((U \cap B) \notin I\). This implies that \(x \in A^a\) or \(x \in B^a\), that is, \(x \in A^a \cup B^a\). So we obtain the equality.

(8) Since \(A-B \subset A\), by (1), \((A-B)^a \subset A^a\) and hence \((A-B)^a - B^a \subset A^a - B^a\). Conversely \(A \subset (A-B) \cup B\), by (7), \(A^a \subset (A-B)^a \cup B^a\) and hence \(A^a - B^a \subset (A-B)^a \cup B^a\) - B^a\). Therefore, \(A^a - B^a \subset (A-B)^a - (B^a \cup B^a)\) and so, \(A^a - B^a \subset (A-B)^a - B^a\).

(9) Assume \(U \in \tau^a(x)\) and \(x \in U \cap A^a\). Then \(x \in U\) and \(x \in A^a\). For \(V \in \tau^a(x)\), \(U \cap V \in \tau^a(x)\) [3]. Thus \(V \cap (U \cap A) = (U \cap V) \cap A \notin I\). So \(x \in (U \cap A)^a\). Therefore \(U \cap A^a \subset (U \cap A)^a\). Also \(U \cap A^a \subset U \cap (U \cap A)^a\), since \(A \cap U \subset A\). Then by (1), \((A \cap U)^a \subset A^a\) and \(U \cap (A \cap U)^a \subset U \cap A^a\). So we get the result.

(10) By (7) and Remark 2(2) \((A \cup U)^a = A^a \cup U^a = A^a \cup \emptyset = A^a\), since \(A - U \subset A\) by (1), \((A-U)^a \subset (A)^a\). So, we get the result.
**Theorem 2.** Let \((X, \tau)\) a topological space, \(\mathcal{I}_1\) and \(\mathcal{I}_2\) be ideals on \(X\) and let \(A\) be a subset of \(X\). Then the following properties hold:

1. If \(\mathcal{I}_1 \subset \mathcal{I}_2\), then \(A^*(\mathcal{I}_1) \subset A^*(\mathcal{I}_2)\);
2. \(A^*(\mathcal{I}_1 \cap \mathcal{I}_2) = A^*(\mathcal{I}_1) \cup A^*(\mathcal{I}_2)\).

**Proof.** (1) Let \(\mathcal{I}_1 \subset \mathcal{I}_2\) and \(x \in A^*(\mathcal{I}_2)\). Then \(A \cap U \notin \mathcal{I}_2\) for every \(U \in \tau^a(x)\) and hence \(A \cap U \notin \mathcal{I}_1\). Then \(x \in A^*(\mathcal{I}_1)\). This shows that \(A^*(\mathcal{I}_2) \subset A^*(\mathcal{I}_1)\).

(2) Since \(\mathcal{I}_1 \cap \mathcal{I}_2 \subset \mathcal{I}_1\) and \(\mathcal{I}_1 \cap \mathcal{I}_2 \subset \mathcal{I}_2\), by Theorem 2 (1) we have. \(A^*(\mathcal{I}_1) \subset A^*(\mathcal{I}_1 \cap \mathcal{I}_2)\) and \(A^*(\mathcal{I}_2) \subset A^*(\mathcal{I}_1 \cap \mathcal{I}_2)\). Hence we have \(A^*(\mathcal{I}_1) \cup A^*(\mathcal{I}_2) \subset A^*(\mathcal{I}_1 \cap \mathcal{I}_2)\). Conversely let \(x \in A^*(\mathcal{I}_1 \cap \mathcal{I}_2)\). Then for every \(U \in \tau^a(x), U \cap A \notin \mathcal{I}_1 \cap \mathcal{I}_2\) hence \(U \cap A \notin \mathcal{I}_1\) or \(U \cap A \notin \mathcal{I}_2\). This shows that \(x \in A^*(\mathcal{I}_1)\) or \(x \in A^*(\mathcal{I}_2)\) and \(x \in A^*(\mathcal{I}_1 \cap \mathcal{I}_2)\). So, we get the result.

**Lemma 2.** Let \((X, \tau, \mathcal{I})\) be an ideal ideal topological space. If \(U \in \tau^a(x)\), then \(U \cap A^* = U \cap (U \cap A)^a \subseteq (U \cap A)^a\) for any subset \(A\) of \(X\).

**Proof.** Suppose that \(U \in \tau^a(x)\) and \(x \in U \cap A^*\). Then \(x \in U\) and \(x \in A^*\). Let \(V\) be any \(a\)-open set containing \(x\). Then \(V \cap U \in \tau^a(x)\) and \(V \cap (U \cap A) = (V \cap U) \cap A \notin \mathcal{I}\). This shows that \(x \in (U \cap A)^a\) and hence we obtain \(U \cap A^a \subseteq (U \cap A)^a\). Moreover, \(U \cap A^a \subseteq U \cap (U \cap A)^a\) and by Theorem 1 \((U \cap A)^a \subseteq A^a\) and \(U \cap (U \cap A)^a \subseteq U \cap A^a\). Therefore, \(U \cap A^a = U \cap (U \cap A)^a\).

4. The open sets of \(\tau^a\)

In this section we have investigated \(\tau^a\) finer than \(\tau\) in the term of the closure operator \(aCl^*(A) = A \cup A^*\). A basis \(\beta(\mathcal{I}, \tau)\) for \(\tau^a\) can be described as follows: A subset \(A\) of an ideal space \((X, \mathcal{I}, \tau)\) is said to be \(\tau^a\)-closed if \(A^a \subset A\). Thus we have \(U \in \tau^a\) if and only if \(X-U\) is \(\tau^a\)-closed which implies \((X-U)^a \subset (X-U)\) and hence \(U \subset X-(X-U)^a\). Thus if \(x \in U\), \(x \notin (X-U)^a\), then there exist \(V \in \tau^a(x)\) such that \(V \cap (X-U) \notin \mathcal{I}\). Hence, let \(I = V \cap (X-U)\) and we have \(x \in V-I \subset U\) where \(V \in \tau^a(x)\) and \(I \in \mathcal{I}\). So the basis for \(\tau^a\) is \(\beta(\mathcal{I}, \tau) = \{V-I : V \in \tau^a(x), I \in \mathcal{I}\}\) and \(\beta\) is not, in general, a topology. See Theorem 4.

**Theorem 3.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space, \(aCl^*(A) = A^a \cup A\) and \(A, B\) be subsets of \(X\). Then

1. \(aCl^*(\emptyset) = \emptyset\).
2. \(A \subseteq aCl^*(A)\).
(3) \(aCl^*(A \cup B) = aCl^*(A) \cup aCl^*(B)\).

(4) \(aCl^*(A) = aCl^*(aCl^*(A))\).

**Proof.** By Theorem 1, we obtain

(1) \(aCl^*(\emptyset) = (\emptyset)^* \cup \emptyset = \emptyset\).

(2) \(A \subseteq A \cup A^* = aCl^*(A)\).

(3) \(aCl^*(A \cup B) = (A \cup B)^* \cup (A \cup B) = (A^* \cup B^*) \cup (A \cup B) = aCl^*(A) \cup aCl^*(B)\).

(4) \(aCl^*(aCl^*(A)) = aCl^*(A^* \cup A) = (A^* \cup A)^* \cup (A^* \cup A) = ((A^*)^* \cup A^*) \cup (A^* \cup A) = A^* \cup A = aCl^*(A)\).

\[\square\]

**Lemma 3.** Let \((X, \tau, I)\) be an ideal topological space and \(A, B\) be subsets of \(X\). Then \(A^* - B^* = (A - B)^* - B^*\).

**Proof.** We have by Theorem 1 \(A^* = [(A - B) \cup (A \cap B)]^* = (A - B)^* \cup (A \cap B)^* \subseteq (A - B)^* \cup B^*\). Thus \(A^* - B^* \subseteq (A - B)^* - B^*\). By Theorem 1, \((A - B)^* \subseteq A^*\) and hence \((A - B)^* - B^* \subseteq A^* - B^*\). Hence \(A^* - B^* = (A - B)^* - B^*\).

\[\square\]

**Lemma 4.** Let \((X, \tau, I)\) be an ideal topological space and \(A, B\) be subsets of \(X\). Then

(1) If \(A \subseteq B\), then \(aCl^*(A) \subseteq aCl^*(B)\).

(2) \(aCl^*(A \cap B) \subseteq aCl^*(A) \cap aCl^*(B)\).

(3) If \(U\) is \(a\)-open, then \(U \cap aCl^*(A) \subseteq aCl^*(U \cap A)\).

**Proof.** (1) Since \(A \subseteq B\), by Theorem 1 we have \(aCl^*(A) = A \cup A^* \subseteq B \cup B^* = aCl^*(B)\).

(2) This is obvious by (1).

(3) Since \(U\) is \(a\)-open, by Lemma 2 we have \(U \cap aCl^*(A) = U \cap (A \cup A^*) = (U \cap A) \cup (U \cap A^*) \subseteq (U \cap A) \cup (U \cap A)^* = aCl^*(U \cap A)\).

\[\square\]

**Theorem 4.** Let \((X, I, \tau)\) be an ideal topological space. Then \(\beta(I, \tau)\) is a basis for \(\tau^a\).

**Proof.** Since \(\emptyset \in I\), Then \(V - \emptyset = V \in \tau^a(x)\) and \(\tau^a(x) \subseteq \beta\) from which it follows that \(X = \cup \beta\) (recall that a - open sets forms a topology). Also \(\beta_1, \beta_2 \in \beta\), and \(I_1, I_2 \in I\), we have \(\beta_1 = V_1 - I_1\) and \(\beta_2 = V_2 - I_2\), where \(V_1, V_2 \in \tau^a(x)\). Then \(\beta_1 \cap \beta_2 = (V_1 - I_1) \cap (V_2 - I_2) = (V_1 \cap (X - I_1)) \cap (V_2 \cap (X - I_2)) = (V_1 \cap V_2) - (I_1 \cup I_2) \in \beta\), where \(V_1 \cap V_2 \in \tau^a(x)\), \(I_1 \cup I_2 \in I\).

\[\square\]

**Remark 3.** The topology \(\tau^a\) finer than \(\tau^a\). See the following example.
Example 2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$, $I = \{\emptyset, \{b\}\}$. Set $A = \{a, c\}$. Then $A \in \tau^a$, but $A$ is not $a$-open. So $A \notin \tau^a(x)$.

Example 3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$, $I = \{\emptyset, \{b\}\}$. Set $A = \{a, c, d\}$. Then $A \notin \tau^a$, but $A \notin \tau^a(x)$.

The following examples show that $\beta(I, \tau)$ is not a topology in general.

Example 4. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\}$, $I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$ be ideal in $X$, where $Int_\delta(A) = \{c, d\}$ is the union of all regular open set of $X$ contained in $A$ and $\{\emptyset, X, \{c, d\}\} \in \tau^a$. Consider the collection of subsets of $X$ defined as $\beta(I, \tau) = \{V - I : V \in \tau^a(x), I \in I\} = \{\emptyset, X, \{c\}, \{d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Thus $\beta(I, \tau)$ is not open under union of any collection of open sets (i.e $\{c\} \cup \{d\} \notin \beta(I, \tau)$) and hence it is not a topology.

5. $a$-compatible topology with an ideal

Definition 2. Let $(X, \tau, I)$ be an ideal topological space. Then $\tau$ is said to be $a$-compatible with respect to $I$, denoted by $\tau \sim^a I$ if and only if, for every $x \in A$ there exist $U \in \tau^a(x)$ such that $U \cap A \in I$, then $A \in I$.

Theorem 5. Let $(X, \tau, I)$ be an ideal topological space and $A$ subset of $X$. Then the following are equivalent:

1. $\tau \sim^a I$,
2. If a subset $A$ of $X$ has a cover $a$- open sets of whose intersection with $A$ is in $I$, then $A$ is in $I$, in other words $A^a = \emptyset$, then $A \in I$,
3. For every $A \subset X$, if $A \cap A^a = \emptyset$, $A \in I$,
4. For every $A \subset X$, $A - A^a \in I$,
5. For every $A \subset X$, if $A$ contains no nonempty subset $B$ with $B \subset B^a$, then $A \in I$.

Proof. (1) $\Rightarrow$ (2) The proof is obvious.

(2) $\Rightarrow$ (3) Let $A \subset X$ and let $x \in A$. Then $x \notin A^a$ and there exists $U_x \in \tau^a(x)$ such that $U_x \cap A \in I$. Since $A$ has a cover $A \subset \cup \{U_x : x \in A\}$ and $U_x \in \tau^a(x)$ by (2), $A \in I$.
(3) ⇒ (4) For any $A \subseteq X$, since $A \cap A^a = \emptyset$, then $A-A^a \subseteq A$ and by Theorem 1(1) $(A-A^a)^a \subseteq A^a$ and $(A-A^a)^a \cap (A-A^a) \subseteq (A-A^a)^a \cap A^a = \emptyset$. Then by (3) we have $A-A^a \in \mathcal{I}$.

(4) ⇒ (5) By (4), for any $A \subseteq X$, $A-A^a \in \mathcal{I}$. $A = (A-A^a) \cup (A \cap A^a)$ and by Theorem 1(4), $A^a = (A-A^a)^a \cup (A \cap A^a)^a = (A \cap A^a)^a$. Therefore, we have $A^a \cap A = (A \cap A^a)^a \cap A$, then $A^a \cap A \subseteq (A \cap A^a)^a$, and $(A \cap A^a) \subseteq A$. By assumption $(A \cap A^a) = \emptyset$. So $A$ contains no nonempty subset. Hence $A-A^a = A$ by (4) $A \in \mathcal{I}$.

(5) ⇒ (1) Let $A \subseteq X$ and assume that for every $a \in A$, there exists $U \in \tau^a(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A^a = \emptyset$. Since $(A-A^a)^a \cap (A-A^a) \subseteq (A-A^a) \cap A^a = \emptyset$. So, $A-A^a$ contains no nonempty subset $B$ with $B \subseteq B^a$. By (5), $A-A^a \in \mathcal{I}$ and hence $A = A \cap (X-A^a) = A-A^a \in \mathcal{I}$.

**Theorem 6.** Let $(X, \tau, \mathcal{I})$ be an ideal ideal topological space. If $\tau$ is $a$-compatible with $\mathcal{I}$, then the following properties are equivalent:

1. For every $A \subseteq X$, $A \cap A^a = \emptyset$ implies that $A^a = \emptyset$;
2. For every $A \subseteq X$, $(A-A^a)^a = \emptyset$;
3. For every $A \subseteq X$, $(A \cap A^a)^a = A^a$.

**Proof.** First, we show that (1) holds if $\tau$ is $a$-compatible with $\mathcal{I}$. Let $A$ be any subset of $X$ and $A \cap A^a = \emptyset$. By Theorem 5, $A \in \mathcal{I}$ and by Remark 2(3) $A^a = \emptyset$.

(1) ⇒ (2) Assume that for every $A \subseteq X$, $A \cap A^a = \emptyset$ implies that $A^a = \emptyset$. Let $B = A-A^a$, then

$$B \cap B^a = (A-A^a) \cap (A-A^a)^a = (A \cap (X-A^a)) \cap (A \cap (X-A^a))^a \subseteq [A \cap (X-A^a)] \cap [A^a \cap (X-A^a)^a] = \emptyset.$$ 

By (1), we have $B^a = \emptyset$. Hence $(A-A^a)^a = \emptyset$.

(2) ⇒ (3) Assume for every $A \subseteq X$, $(A-A^a)^a = \emptyset$.

$$A = (A-A^a) \cup (A \cap A^a) \quad \text{and} \quad A^a = [(A-A^a) \cup (A \cap A^a)]^a = (A-A^a)^a \cup (A \cap A^a)^a = (A \cap A^a)^a.$$ 

(3) ⇒ (1) Assume for every $A \subseteq X$, $A \cap A^a = \emptyset$ and $(A \cap A^a)^a = A^a$. This implies that $\emptyset = \emptyset^a = A^a$.■
Theorem 7. Let \((X, \tau, I)\) be an ideal topological space, then the following properties are equivalent:

1. \(\tau^a \cap I = \emptyset\);
2. If \(I \in I\), then \(\text{aInt}(I) = \emptyset\);
3. For every \(G \in \tau^a\), \(G \subseteq G^a^\ast\);
4. \(X = X^a^\ast\).

Proof. (1) \(\Rightarrow\) (2) Let \(\tau^a \cap I = \emptyset\) and \(I \in I\). Suppose that \(x \in \text{aInt}(I)\). Then there exists \(U \in \tau^a\) such that \(x \in U \subseteq I\). Since \(I \in I\) and hence \(\emptyset \neq \{x\} \subseteq U \in \tau^a \cap I\). This is contrary that \(\tau^a \cap I = \emptyset\). Therefore, \(\text{aInt}(I) = \emptyset\).

(2) \(\Rightarrow\) (3) Let \(x \in G\). Assume \(x \notin G^a^\ast\) then there exists \(U_x \in \tau^a(x)\) such that \(G \cap U_x \in I\). By (2), \(x \in G \cap U_x = \text{aInt}(G \cap U_x) = \emptyset\). Hence \(x \in G^a^\ast\) and \(G \subseteq G^a^\ast\).

(3) \(\Rightarrow\) (4) Since \(X\) is a-open, then \(X = X_+\).

(4) \(\Rightarrow\) (1) \(X = X^a^\ast\) = \{\(x \in X : U \cap X = U \notin I\) for each a-open set \(U\) containing \(x\). Hence \(\tau^a \cap I = \emptyset\).

\[\blacksquare\]

Theorem 8. Let \((X, \tau, I)\) be an ideal topological space and \(\tau\) be a-compatible with \(I\). Then for every \(G \in \tau^a\) and any subset \(A\) of \(X\), \((G \cap A)^a^\ast = (G \cap A^a^\ast)^a^\ast = \text{aCl}(G \cap A^a^\ast)\).

Proof. (1) Let \(G \in \tau^a\). Then by Lemma 2, \(G \cap A^a^\ast = G \cap (G \cap A)^a^\ast \subseteq (G \cap A)^a^\ast\) and hence \((G \cap A)^a^\ast)^a^\ast \subseteq ((G \cap A)^a^\ast)^a^\ast \subseteq (G \cap A)^a^\ast\) by Theorem 1.

(2) Now by using Theorem 1 and Theorem 6, we obtain \((G \cap (A - A^a^\ast))^a^\ast \subseteq (G \cap A - (A - A^a^\ast))^a^\ast = G^a^\ast \cap \emptyset = \emptyset\). Moreover, \((G \cap A)^a^\ast - (G \cap A^a^\ast)^a^\ast \subseteq (G \cap A) - (G \cap A^a^\ast)^a^\ast = (G \cap (A - A^a^\ast))^a^\ast = \emptyset\), which implies that \((G \cap A)^a^\ast \subseteq (G \cap A^a^\ast)^a^\ast\). By (1) and (2), we obtain \((G \cap A)^a^\ast = (G \cap A^a^\ast)^a^\ast\).

By Theorem 1, \((G \cap A)^a^\ast = (G \cap A^a^\ast)^a^\ast \subseteq \text{aCl}(G \cap A^a^\ast)\). Also, in view of Lemma 2, we have \(G \cap A^a^\ast \subseteq (G \cap A)^a^\ast\) and hence \(\text{aCl}(G \cap A^a^\ast) \subseteq \text{aCl}((G \cap A)^a^\ast)\). Consequently, we obtain \((G \cap A^a^\ast)^a^\ast = (G \cap A)^a^\ast = \text{aCl}(G \cap A^a^\ast)\).

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