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SOLVABILITY OF SEQUENCE SPACES EQUATIONS
OF THE FORM \((E_a)_\Delta + F_x = F_b\)

Abstract. Given any sequence \(a = (a_n)_{n \geq 1}\) of positive real numbers and any set \(E\) of complex sequences, we write \(E_a\) for the set of all sequences \(y = (y_n)_{n \geq 1}\) such that \(y/a = (y_n/a_n)_{n \geq 1} \in E\); in particular, \(s_a^{(c)}\) denotes the set of all sequences \(y\) such that \(y/a\) converges. For any linear space \(F\) of sequences, we have \(F_x = F_b\) if and only if \(x/b\) and \(b/x\) belong to \(M(F,F)\). The question is: what happens when we consider the perturbed equation \((E_a)_\Delta + F_x = F_b\) where \(E\) is a special linear space of sequences? In this paper we deal with the perturbed sequence spaces equations (SSE), defined by \((E_a)_\Delta + s^{(c)}_X = s^{(c)}_B\) where \(E = c_0\), or \(\ell_p\), \((p > 1)\) and \(\Delta\) is the operator of the first difference defined by \(\Delta_n y = y_n - y_{n-1}\) for all \(n \geq 1\) with the convention \(y_0 = 0\). For \(E = c_0\) the previous perturbed equation consists in determining the set of all positive sequences \(x = (x_n)_n\) that satisfy the next statement. The condition \(y_n/b_n \rightarrow L_1\) holds if and only if there are two sequences \(u, v\) with \(y = u + v\) such that \(\Delta_n u/a_n \rightarrow 0\) and \(v_n/x_n \rightarrow L_2\) \((n \rightarrow \infty)\) for all \(y\) and for some scalars \(L_1\) and \(L_2\). Then we deal with the resolution of the equation \((E_a)_\Delta + s^{(c)}_X = s^{(c)}_B\) for \(E = c\), or \(s_1\), and give applications to particular classes of (SSE).

Key words: BK space, spaces of strongly bounded sequences, sequence spaces equations, sequence spaces equations with operator.

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1. Introduction

We write \(\omega\) for the set of all complex sequences \(y = (y_n)_{n \geq 1}\), \(\ell_\infty\), \(c\) and \(c_0\) for the sets of all bounded, convergent and null sequences, respectively, also \(\ell^p = \{y \in \omega : \sum_{k=1}^{\infty} |y_k|^p < \infty\}\) for \(1 \leq p < \infty\). If \(y, z \in \omega\), then we write \(y z = (y_n z_n)_{n \geq 1}\). Let \(U = \{y \in \omega : y_n \neq 0\}\) and \(U^+ = \{y \in \omega : y_n > 0\}\). We write \(z/u = (z_n/u_n)_{n \geq 1}\) for all \(z \in \omega\) and all \(u \in U\), in particular
$1/u = e/u$, where $e = 1$ is the sequence with $e_n = 1$ for all $n$. Finally, if $a \in U^+$ and $E$ is any subset of $\omega$, then we put $E_a = (1/a)^{-1} \ast E = \{y \in \omega : y/a \in E\}$. Let $E$ and $F$ be subsets of $\omega$. Then the set $M(E,F) = \{y \in \omega : yz \in F \text{ for all } z \in E\}$ is called the multiplier space of $E$ and $F$. 

In [1], the sets $s_a$, $s_a^0$ and $s_a^{(c)}$ were defined for positive sequences $a$ by $(1/a)^{-1} \ast E$ and $E = \ell_{\infty}, c_0, c,$ respectively. In [3] the sum $E_a + F_b$ and the product $E_a \ast F_b$ were defined where $E, F$ are any of the symbols $s, s^0, s^{(c)}$. Then in [6] the solvability was determined of sequences spaces inclusion equations $G_b \subset E_a + F_b$ where $E, F, G \in \{s^0, s^{(c)}, s\}$ and some applications were given to sequence spaces inclusions with operators.

In this paper we deal with the solvability of perturbed equations defined as follows. Let $F$ be any linear space of sequences, and $b$ be a positive sequence. It is known that the solutions of the equation $F_x = F_b$ where $x$ is the unknown, are determined by $x \in cl^{M(F,F)}(b)$. Then we consider the perturbed equation $\mathcal{E} + F_x = F_b$, where $\mathcal{E}$ is a particular linear space of sequences. For example, the solutions of the equation $c_x = c$ are determined by $\lim_{n \to \infty} x_n = L > 0$. Then the perturbed equation defined by $c_a + c_x = c$, has the same solutions if and only if $a_n \to 0$ as $n$ tends to infinity; then if $a_n \to l > 0$ as $n$ tends to infinity the set of all its solutions is equal to $c$; finally, if $a \notin c$ the perturbed equation has no solutions, (cf. [7]). Here we extend some results given in [12], [6], [4], [5], [11], [7], [9], [10]. In [11] for given sequences $a$ and $b$ was determined the set of all positive sequences $x$ for which $y_n/b_n \to l$ if and only if there are sequences $u$ and $v$ for which $y = u + v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ ($n \to \infty$) for all $y$ and for some scalars $l$ and $l'$. This statement is equivalent to the sequence spaces equation $s_a^0 + s_x^{(c)} = s_b^{(c)}$.

In [7] we determined the set of all $x \in U^+$ such that for every sequence $y$, we have $y_n/b_n \to l$ if and only if there are sequences $u$ and $v$ with $y = u + v$ and $|u_n/a_n|^{1/n} \to 0$ and $v_n/x_n \to l'$ ($n \to \infty$) for some scalars $l$ and $l'$. This statement means $\Gamma_a + s^{(c)}_x = s^{(c)}_b$, where $\Gamma$ is the set of all entire sequences. So we are led to deal with special sequence spaces equations (SSE), (resp. sequence spaces inclusion equations (SSIE)), which are determined by an identity, (resp. inclusion), for which each term is a sum or a sum of products of sets of the form $(E_a)_{T}$ and $(E_{f(x)})_{T}$ where $f$ maps $U^+$ to itself, $E$ is a linear space of sequences, $x$ is the unknown and $T$ is a triangle. It can be found in [5] a solvability of the (SSE) $E_a + \left( s^{(c)}_x \right)_{B(r,s)} = s^{(c)}_x$ where $E = s, s^0,$ or $s^{(c)}$ and $x$ is the unknown. In [11] we determined the sets of all positive sequences $x$ that satisfy each of the systems $s^0_a + (s_x)_\Delta = s_b$, $s_x \supset s_b$ and $s_a + (c_x)_\Delta = c_b$, $c_x \supset c_b$. Then a resolution can be found of the (SSE) with operators defined by $(E_a)_{C(\lambda)}D_\tau + (c_x)_{C(\mu)}D_\tau = c_b$ with $E = c_0$, or $\ell_{\infty}$. Recently in [8] a study can be found on the (SSE) with operator
\((E_a)_{C(\lambda)c} + (E_x)_{C(\lambda)c} = E_b\), where \(b \in \mathcal{C}_1\) and \(E\) is any of the sets \(\ell_\infty\), or \(c_0\). For \(E = c_0\) the resolution of this equation consists in determining the set of all \(x \in U^+\) such that for every sequence \(y\) the condition \(y_n/b_n \to 0\) \((n \to \infty)\) holds if and only if there are \(u, v \in \omega\) such that \(y = u + v\) and

\[
\frac{1}{\lambda_n a_n} \sum_{k=1}^{n} \left( \frac{1}{\mu_k} \sum_{i=1}^{k} u_i \right) \to 0 \quad \text{and} \quad \frac{1}{\lambda_n \sigma_n x_n} \sum_{k=1}^{n} \left( \frac{1}{\mu_k} \sum_{i=1}^{k} v_i \right) \to 0 \quad (n \to \infty).
\]

There is also a resolution of the (SSE) \((s_a)_{(C(\lambda)c)} + (s^0_x)_{(C(\mu)c)} = s^0_b\).

In this paper we deal with some classes of (SSE) with the operators of the form \((E_a)_{\Delta} + s^0_x = s^0_b\) where \(E = c_0\), or \(\ell^p\), \((p > 1)\) and of the form \((E_a)_{\Delta} + s^0_x = s^0_b\).

This paper is organized as follows. In Section 2 we recall some definitions and results on sequence spaces and matrix transformations. In Section 3 we recall general results on the multiplier \(M(E, F)\) of some sequence spaces. In Section 4 we recall some results on the solvability of the equation \(E + F_x = F_b\) in the general case. In Section 5 we determine the sets \(M((E_a)_{\Delta}, F)\) and we deal with the (SSIE) \(F_b \subset (E_a)_{\Delta} + F_x\). In Section 6 we apply the previous results to solve the (SSE) \((E_a)_{\Delta} + c_x = c_b\) where \(E = c_0\), or \(\ell^p\), \((p > 1)\). In Section 7 we apply the results of Section 6 to solve special (SSE) of the form \((E_a)_{\Delta} + c_x = c_b\). Finally in Section 8 we deal with the (SSE) \((E_a)_{\Delta} + s^0_x = s^0_b\) for \(E = c\), or \(s_1\).

## 2. Preliminaries and notations

An FK space is a complete metric space, for which convergence implies coordinatewise convergence. A BK space is a Banach space of sequences that is an FK space. A BK space \(E\) is said to have AK if for every sequence \(y = (y_n)_n \in E\), then \(y = \lim_{p \to \infty} \sum_{k=1}^{p} y_k e^{(k)}\), where \(e^{(k)} = (0, \ldots, 1, \ldots)\), 1 being in the \(k\)th position.

For any given infinite matrix \(A = (a_{nk})_{n,k}\) we define the operators \(A_n = (a_{nk})_{k \geq 1}\) for any integer \(n \geq 1\), by \(A_n y = \sum_{k=1}^{\infty} a_{nk} y_k\), where \(y = (y_k)_{k \geq 1}\), and the series are assumed convergent for all \(n\). So we are led to the study of the operator \(A\) defined by \(Ay = (A_n y)_n\) mapping between sequence spaces. When \(A\) maps \(E\) into \(F\), where \(E\) and \(F\) are subsets of \(\omega\), we write \(A \in (E, F)\), (cf. [13]). It is well known that if \(E\) has AK, then the set \(\mathcal{B}(E)\) of all bounded linear operators \(L\) mapping in \(E\), with norm \(\|L\| = \sup_{y \neq 0} (\|L(y)\|_E / \|y\|_E)\) satisfies the identity \(\mathcal{B}(E) = (E, E)\). We will use the operator \(\Delta\) of the first difference defined by \(\Delta_n y = y_n - y_{n-1}\)
for \( n \geq 1 \) with the convention \( y_0 = 0 \). It is well known that the operator \( A \) defined by \( y_k = \sum_{k=1}^{n} y_k \) for all \( n \), is the inverse of \( \Delta \), that is, \( \Delta (\Sigma y) = \Sigma (\Delta y) = y \) for all \( y \). Let \( U^+ \subset \omega \) be the set of all sequences \( u = (u_n)_n \) with \( u_n > 0 \) for all \( n \). Then for any given sequence \( u = (u_n)_n \in \omega \) we define the infinite diagonal matrix \( D_u \) by \( [D_u]_{nn} = u_n \) for all \( n \). It is interesting to rewrite the set \( E_u \) using a diagonal matrix. Let \( E \) be any subset of \( \omega \) and \( u \in U^+ \) we have

\[
E_u = D_uE = \{ y = (y_n)_{n \geq 1} \in \omega : y/u \in E \}.
\]

We will use the sets \( s^0_a, s^c_a, s_a \) and \( \ell^0_a \) defined as follows (cf. [1]). For given \( a \in U^+ \) and \( p \geq 1 \) we put \( D_ac_0 = s^0_a, D_ac = s^c_a, D_a\ell = s_a, \) and \( D_a\ell_p = \ell^0_a \).

Each of the spaces \( D_aE \), where \( E \in \{c_0, c, \ell, \ell_p\} \) with \( p > 1 \), is a BK space, and we have \( \|y\|_{s_a} = \sup_{n \geq 1} (|y_n|/a_n) \), and \( \|y\|_{e^p_a} = (\sum_{k=1}^{\infty} (|y_k|/a_k)^p)^{1/p} \). Then and \( s^0_a \) and \( \ell^0_a \) have AK. If \( a = (r^n)_n \) with \( r > 0 \), we write \( s_r, s^r_a \) and \( s^c_r \) for the sets \( s_a, s^0_a \) and \( s^c_a \) respectively. When \( r = 1 \), we obtain \( s_1 = \ell, s^1_0 = c_0 \) and \( s^1_1 = c \). Recall that \( S_1 = (s_1, s_1) \) is a Banach algebra and \( (c_0, s_1) = (c, \ell) = (s_1, s_1) = S_1 \). We have \( A \in S_1 \) if and only if

\[
(2) \quad \sup_n \left( \sum_{k=1}^{\infty} |a_{nk}| \right) < \infty.
\]

We will also use the well-known characterizations of \( (c_0, c_0) \) and \( (c_0, c) \). We have \( A \in (c_0, c_0) \) if and only if (2) holds and \( \lim_{n \to \infty} a_{nk} = 0 \) for all \( k \); and we have \( A \in (c_0, c) \) if and only if (2) holds and \( \lim_{n \to \infty} a_{nk} = l_k \) for some scalar \( l_k \) and for all \( k \). For any subset \( F \) of \( \omega \), we write \( F_A = \{ y \in \omega : Ay \in F \} \). Let \( cs = c \Sigma \) denote the set of all convergent series. For any subset \( E \) of \( \omega \) we write \( AE = \{ y \in \omega : y = Ax \) for some \( x \in \omega \} \). We will use the well-known property, stated as follows. For any given triangles \( T \) and \( T' \), we have \( T' \in (E, F) \) if and only if \( T'T^{-1} \in (E, F) \) for any subsets \( E, F \subset \omega \). It is also well known that \( A \in (E, F_T) \) if and only if \( TA \in (E, F) \).

3. The multipliers of some sets and matrix transformations

3.1. The multipliers of classical sets

First we need to recall some well known results. Let \( y \) and \( z \) be sequences and let \( E \) and \( F \) be two subsets of \( \omega \), we then write \( yz = (y_nz_n)_{n \geq 1} \) and

\[
M (E, F) = \{ y \in \omega : yz \in F \text{ for all } z \in E \}.
\]

\( M (E, F) \) is called the multiplier space of \( E \) and \( F \). In the following we will use the next elementary results.
Lemma 1. Let $E$, $\tilde{E}$, $F$ and $\tilde{F}$ be arbitrary subsets of $\omega$. Then

(i) $M (E, F) \subset M \left(\tilde{E}, \tilde{F}\right)$ for all $\tilde{E} \subset E$,

(ii) $M (E, F) \subset M \left(E, \tilde{F}\right)$ for all $F \subset \tilde{F}$.

Lemma 2. Let $a, b \in U^+$ and let $E$ and $F$ be two subsets of $\omega$. Then $D_a E \subset D_b F$ if and only if $a/b \in M (E, F)$.

Lemma 3. Let $a, b \in U^+$ and let $E$, $F \subset \omega$. Then $A \in (D_a E, D_b F)$ if and only if $D_{1/b} A D_a \in (E, F)$.

By [14, Lemma 3.1, p. 648] and [16, Example 1.28, p. 157], we obtain the next result.

Lemma 4. We have

(i) $M (c, c_0) = M (\ell_\infty, c) = M (\ell_\infty, c_0) = c_0$ and $M (c, c) = c$;

(ii) $M (\chi, \ell_\infty) = M (c_0, \chi') = \ell_\infty$ for $\chi, \chi' = c_0, c$, or $\ell_\infty$.

4. On the (SSE) $E + F_x = F_b$

In this section we apply the previous results to the solvability of the (SSE) $E + F_x = F_b$ with $1 \in F$.

4.1. Regular sequence spaces equations

For $b \in U^+$ and for any subset $F$ of $\omega$, we denote by $cl^F (b)$ the equivalent class for the equivalence relation $R_F$ defined by

$$x R_F y \text{ if } D_x F = D_y F \text{ for } x, y \in U^+.$$  

It can easily be seen that $cl^F (b)$ is the set of all $x \in U^+$ such that $x/b \in M (F, F)$ and $b/x \in M (F, F)$, (cf. [11]). We then have $cl^F (b) = cl^{M(F,F)} (b)$. For instance $cl^c (b)$ is the set of all $x \in U^+$ such that $D_x c = D_b c$, that is, $s^{(c)}_x = s^{(c)}_b$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ ($n \to \infty$) for some $C > 0$. In [11] we denote by $cl^\infty (b)$ the class $cl^{\ell_\infty} (b)$. Recall that $cl^\infty (b)$ is the set of all $x \in U^+$ such that $K_1 \leq x_n/b_n \leq K_2$ for all $n$ and for some $K_1, K_2 > 0$.

Let $X$ and $Y$ be two linear spaces of sequences. Then the sum of $X$ and $Y$ defined by $Z = X + Y = \{x + y : x \in X \text{ and } y \in Y\}$, is a linear space of sequences. Let $b$ be a positive sequence and $F$ be a linear subspace of $\omega$. As we have seen above the solutions of the equation $F_x = F_b$ are defined by $x \in cl^F (b)$. Then the question is: what are the solutions of the perturbed equation $E + F_x = F_b$, where $E$ is a linear space of sequences? In this way we are led to consider the set $S (E, F) = \{x \in U^+ : E + F_x = F_b\}$, where $b \in U^+$, and $E$ is a linear subspace of $\omega$. 
**Definition 1.** We say that $S(E, F)$, (or the equation $E + F_x = F_b$), is regular if

$$S(E, F) = \begin{cases} \text{cl}M(F,F)(b), & \text{if } 1/b \in M(E, F), \\ \emptyset, & \text{if } 1/b \notin M(E, F). \end{cases}$$

Note that $E + F_x = F_b$ is not regular in general. Indeed for $E = F = \ell_\infty$ we have $M(\ell_\infty, \ell_\infty) = \ell_\infty$ and if $1/b \in \ell_\infty \backslash c_0$ and $s_a = s_1$ we have $S(\ell_\infty, \ell_\infty) = s_b \cap U^+ \neq \text{cl}^{\infty}(b)$, (cf. [12, Theorem 11, pp. 916-917]).

In particular the solutions of the (SSE) $\ell_\infty + s_x = \ell_\infty$ are determined by $0 < x_n \leq M$ for all $n$ and for some $M > 0$. It is interesting to notice that by [7, Theorem 5.2, p. 108], the (SSE) $c + c_x = c_b$ is not regular, since $1/b \in c \backslash c_0$ implies $S(c, c) = c_b$.

In the following we will use the condition

$$(3) \quad \chi \subset \chi(D_\alpha) \text{ for all } \alpha \in c(1),$$

where $\chi \subset \omega$ is any linear space, and $c(1)$ is the set of all sequences that tend to 1. It can easily seen that this condition is true for any of the spaces $F = c$, $s_1$. To state the next results we also need the next conditions:

$$(4) \quad 1 \in F,$$

$$(5) \quad F \subset M(F,F).$$

We then recall the next result which is a direct consequence of [7, Theorem 5.1, pp. 106-107].

**Lemma 5.** Let $b \in U^+$ and let $E$, $F$ be two linear subspaces of $\omega$. We assume $F$ satisfies the conditions in (3), (4), (5), and that $M(E, F) \subset M(E, c_0)$. Then the set $S(E, F)$ is regular.

5. Some results on the multiplier $M((E_a)_\Delta, F)$ and on the (SSIE) $F_b \subset (E_a)_\Delta + F_x$

In this section we explicitly calculate the multiplier $M((E_a)_\Delta, F)$ where $E = c_0$, or $\ell_p$ and $F = c_0$, $c$, or $s_1$. Then we deal with the (SSIE) defined by $F_b \subset (E_a)_\Delta + F_x$ where $E$ and $F$ are linear spaces of sequences, and $E \subset s_1$ and $c_0 \subset F \subset s_1$.

In the following we will use the factorable matrix $D_\alpha \Sigma D_\beta$, with $\alpha$ and $\beta \in \omega$ defined by $(D_\alpha \Sigma D_\beta)_{nk} = \alpha_n \beta_k$ for $k \leq n$ for all $n$, the other entries being equal to zero.
5.1. The multipliers $M \left( (E_a)_\Delta, F \right)$ where $E = c_0$, or $\ell_p$
and $F = c_0$, $c$, or $s_1$

Lemma 6. Let $a \in U^+$ and let $p > 1$. Then
(i) the condition $a \notin cs$ implies

$$M \left( (s_0^a)_\Delta, F \right) = s \left( \frac{1}{\sum_{k=1}^{n} a_k} \right)_n$$

for $F = c_0$, $c$, or $s_1$.

(ii) The condition $a^q \notin cs$ where $q = p/(p-1)$ implies

$$M \left( (\ell_p^a)_\Delta, F \right) = \left( \frac{1}{\sum_{k=1}^{n} a_k^{q-1}} \right)_n$$

for $F = c_0$, $c$, or $s_1$.

Proof. (i) We have $\alpha \in M \left( (s_0^a)_\Delta, c_0 \right)$ if and only if $D_\alpha \Sigma D_a \in (c_0, c_0)$. By the characterization of $(c_0, c_0)$ we have

$$|\alpha_n| \sum_{k=1}^{n} a_k \leq K \text{ for all } n \text{ and for some } K > 0$$

and

$$\alpha \in c_0.$$

But since $a \notin cs$ the condition in (8) implies (9) and we have $\alpha \in M \left( (s_0^a)_\Delta, c_0 \right)$ if and only if (8) holds. This shows the identity in (6) for $F = c_0$. In a similar way the identity (6) for $F = s_1$ can easily be shown. From the inclusions

$$M \left( (s_0^a)_\Delta, c_0 \right) \subset M \left( (s_0^a)_\Delta, c \right) \subset M \left( (s_0^a)_\Delta, s_1 \right),$$

we conclude that the identity in (6) holds for $F = c_0$.

(ii) We have $\alpha \in M \left( (\ell_p^a)_\Delta, c_0 \right)$ if and only if $D_\alpha \Sigma D_a \in (\ell_p, c_0)$. By the characterization of $(\ell_p, c_0)$, (see for instance [16, Theorem 1.37, pp. 160-161]), we have

$$|\alpha_n|^{q} \sum_{k=1}^{n} a_k^{q} \leq K \text{ for all } n \text{ and for some } K > 0$$

and (9) holds. But since $a^q \notin cs$ the condition in (10) implies (9) and we have $\alpha \in M \left( (\ell_p^a)_\Delta, c_0 \right)$ if and only if (10) holds. So we have shown that the identity in (7) holds for $F = c_0$. In a similar way the identity in (7) with $F = s_1$ can easily be shown. We conclude the proof using the inclusions

$$M \left( (\ell_p^a)_\Delta, c_0 \right) \subset M \left( (\ell_p^a)_\Delta, c \right) \subset M \left( (\ell_p^a)_\Delta, s_1 \right).$$

5.2. Some properties of the (SSIE) $F_b \subset (E_a)_\Delta + F_x$

Let $E$ and $F$ be two linear subspaces of $\omega$. We define by $\mathcal{I} \left( (E_a)_\Delta, F \right)$ the set of all $x \in U^+$ such that $F_b \subset (E_a)_\Delta + F_x$. It can easily be seen that the
sets \((E_a)_\Delta\) and \(F_x\) are linear spaces of sequences, and we have \(z \in (E_a)_\Delta + F_x\) if and only if there are \(\xi \in E\) and \(f \in F\) such that 
\[
z_n = \sum_{k=1}^{n} a_k \xi_k + f_n x_n.
\]
To simplify we will denote by \(I_E^F\) the set \(I((E_a)_\Delta, F)\).

In the following we will use the sequence \(\sigma = (\sigma_n)_n\), defined for \(a, b \in U^+\) by
\[
\sigma_n = \frac{1}{b_n} \sum_{k=1}^{n} a_k.
\]
For any given \(b \in U^+\) we write \(s_b^\bullet\) for the set of all sequences \(x\) such that \(x_n \geq Kb_n\) for all \(n\) and for some \(K > 0\). Notice that we have \(s_b \cap s_b^\bullet = cl^\infty (b)\).

First we state the next lemma.

**Lemma 7.** Let \(a, b \in U^+\), and let \(E\) and \(F\) be two linear subspaces of \(s\), that satisfy \(E, F \subset s_1\) and \(F \supset c_0\). Then we have

(i) Assume \(\sigma \in c_0\). Then

a) \(I_E^F \subset I_{s_1}^1\),

b) \(I_E^F \subset s_b^\bullet\).

(ii) Assume \(a \in c_0\). Then we have \(I_E^F \subset s_1^\bullet\) for \(b = e\).

**Proof.** (i) a) Let \(x \in I_E^F\). Then we have \(F_b \subset (E_a)_\Delta + F_x\) and since \(E, F \subset s_1\) we obtain
\[
(E_a)_\Delta + F_x = \Sigma D_a E + D_x F \subset (\Sigma D_a + D_x) s_1,
\]
where \(\Sigma D_a + D_x\) is a triangle, and

(11) \[
F_b \subset (s_1)_T,
\]
with \(T = (\Sigma D_a + D_x)^{-1}\). Now the condition in (11) implies \(T \in (F_b, s_1)\), but we have, since \(F \supset c_0\)

\[
(F_b, s_1) \subset (s_0^b, s_1) = (s_b, s_1)
\]
and then \(T \in (s_b, s_1)\). Finally we obtain \(s_b \subset (s_a)_\Delta + s_x\). This shows the inclusion \(I_E^F \subset I_{s_1}^1\).

(i) b) As we have just seen \(x \in I_E^F\) implies \(s_b \subset (s_a)_\Delta + s_x\) and there are \(u \in (s_a)_\Delta\) and \(v \in s_x\) such that \(b = u + v\). Since \((s_a)_\Delta = (\Sigma D_a) s_1\) and \(b \in s_b\), there are two sequences \(h, k \in s_1\) such that \(b_n = \sum_{k=1}^{n} a_k h_k + x_n k_n\) and
\[
\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^{n} a_k h_k\right) = k_n \text{ for all } n.
\]
Then we have
\[
\frac{1}{b_n} \sum_{k=1}^{n} a_k h_k \leq K \sigma_n \text{ for all } n \text{ and for some } K > 0,
\]
and since \( \sigma \in c_0 \), we conclude \( b/x \in s_1 \), that is, \( x \in s_b^* \).

(ii) Let \( x \in \mathcal{I}_E^F \). Then from (i) we obtain \( s_1 \subset (s_a)_\Delta + s_x \). So the sequence \( \xi = ((-1)^n)_n \in s_1 \) can be written as \( \xi = u + v \), where \( u \in (s_a)_\Delta \) and \( v \in s_x \). There are \( K_1 \) and \( K_2 > 0 \) such that \( |\Delta_n u| = |u_n - u_{n-1}| \leq K_1 a_n \), \( |\Delta_n v| = |v_n - v_{n-1}| \leq K_2 (x_n + x_{n-1}) \) and

\[
|((\Delta \xi)_n| = 2 = |\Delta_n u + \Delta_n v| \leq K_1 a_n + K_2 (x_n + x_{n-1}) \text{ for all } n \geq 2.
\]

Then we have

\[
x_n + x_{n-1} \geq \frac{1}{K_2} (2 - K_1 a_n),
\]

and since \( a \in c_0 \), there is \( K_3 > 0 \) such that \( x_n + x_{n-1} \geq K_3 \) for all sufficiently large \( n \), and it can easily be shown \( x \in s_1^* \). We conclude \( \mathcal{I}_E^F \subset s_1^* \). This completes the proof. \( \blacksquare \)

### 6. Solvability of sequence spaces equations of the form

\[
(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}
\]

In this section we solve the (SSE) \((E_a)_\Delta + s_x^{(c)} = s_b^{(c)}\) where \( E = c_0 \), or \( \ell^p \) with \( p > 1 \). For instance, the (SSE) defined by \((s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}\) is equivalent to the statement: \( y_n/b_n \to l_1 \) \((n \to \infty)\) if and only if there are two sequences \( u, v \) with \( y = u + v \) such that \((\Delta_n u)/a_n \to 0\) and \( v_n/x_n \to l_2 \) \((n \to \infty)\) for all \( y \) and for some scalars \( l_1 \) and \( l_2 \).

#### 6.1. Solvability of the (SSE) \((s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}\) and \((\ell^p_a)_\Delta + s_x^{(c)} = s_b^{(c)}\) in the general case

For any given \( a, b \in U^+ \) we denote by \( S((s_a^0)_\Delta, F) \) the set of all the solutions of the (SSE) defined by \((E_a)_\Delta + F_x = F_b \) where \( E \) and \( F \) are linear spaces.

**Theorem 1.** Let \( a, b \in U^+ \). Then we have:

(i) The set \( S^c_0 = S\left((s_a^0)_\Delta, c\right) \) of all the solutions of the (SSE) \((s_a^0)_\Delta + s_x^{(c)} = s_b^{(c)}\) is determined in the following way.

a) If \( a \notin cs \) (that is, \( \sum_k a_k = \infty \)), then we have

\[
S^c_0 = \begin{cases} 
cl^c(b), & \text{if } \sigma \in s_1, \\
\varnothing, & \text{if } \sigma \notin s_1.
\end{cases}
\]

b) If \( a \in cs \), then we have

\[
S^c_0 = \begin{cases} 
cl^c(b), & \text{if } \frac{1}{b} \in c_0, \\
cl^c(e), & \text{if } \frac{1}{b} \in c \setminus c_0, \\
\varnothing, & \text{if } \frac{1}{b} \notin c.
\end{cases}
\]

(12)
(ii) The set $S^c_p = S((\ell^p_n)_\Delta,c)$ with $p > 1$, of all the solutions of the (SSE) \((\ell^p_n)_\Delta + s^x_b = s^c_b\) is determined in the following way.

a) If \(a^q \notin c, s\), then

\[
S^c_p = \begin{cases}
cl^c(b), & \text{if } \left(\frac{a_1^q + \ldots + a_n^q}{b_h^q}\right) \in s_1, \\
\emptyset, & \text{if } \left(\frac{a_1^q + \ldots + a_n^q}{b_h^q}\right) \notin s_1.
\end{cases}
\]

b) If \(a^q \in c, s\), then $S^c_p = S^c_0$ defined by (12).

**Proof.** (i) a) First consider the case $a \notin cs$. By Lemma 6 we have $M((s_0^a)_\Delta,c) = M((s_0^a)_\Delta,c_0)$ and we can apply Lemma 5 where $1/b \in M((s_0^a)_\Delta,c)$ if and only if $\sigma \in s_1$.

(i) b) Case $a \in cs$. We deal with the 3 cases $\alpha)$ $1/b \notin c, \beta)$ $1/b \in c_0$ and $\gamma)$ $1/b \in c \setminus c_0$.

Case $\alpha)$. We have $S^c_0 = \emptyset$. Indeed, assume there is $x \in S^c_0$, then we have $(s_0^a)_\Delta \subset s^c_b$ and $D_{1/b}\Sigma D_a \in (c_0,c)$. From the characterization of $(c_0,c)$ we deduce $1/b \in c$, which is a contradiction. We conclude $S^c_0 = \emptyset$.

Case $\beta)$. Let $1/b \in c_0$. Then $x \in S^c_0$ implies

\[(13) \quad x \in s^c_b\]

and $s^c_b \subset (s_0^a)_\Delta + s^c_x$. Using similar arguments as those in Lemma 7, we easily see that since $b \in s^c_b$ there are $\varepsilon \in c_0$ and $\varphi \in c$ such that

\[
\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^{n} a_k \varepsilon_k\right) = \varphi_n \text{ for all } n.
\]

We deduce $b/x \in c$ since $\sigma \in c_0$. Using the condition in (13) we conclude that $x \in S^c_0$ implies $s^c_x = s^c_b$. Conversely, assume $s^c_x = s^c_b$. Then we have

\[(s_0^a)_\Delta + s^c_x = (s_0^a)_\Delta + s^c_b = s^c_b\]

since we have $\sigma \in s_1$ and $1/b \in c$. We conclude $S^c_0 = cl^c(b)$.

Case $\gamma)$. Here we have $\lim_{n \to \infty} b_n = L > 0$ and $s^c_b = c$ and we are led to study the (SSE)

\[(14) \quad (s_0^a)_\Delta + s^c_x = c.
\]

We have $x \in S^c_0$ implies $s^c_x \subset c$, that is, $x_n \to l (n \to \infty)$. Then by Lemma 7 (ii) with $E = s_1$ and $F = c$, the condition $x \in S^c_0$ implies $x \in s^*_1$. This means $l > 0$ and $S^c_0 = cl^c(c)$. This completes the proof.
(ii) a) Case $a^q \notin cs$. By Lemma 6 we have $M ((\ell^p_a)_\Delta, c) = M ((\ell^p_a)_\Delta, c_0)$. Let $\alpha \in M ((\ell^p_a)_\Delta, c)$. Then we can apply Lemma 5 where $1/b \in M ((\ell^p_a)_\Delta, c)$ if and only if $(\sum_{k=1}^n a_k^q / b_n^q)_n \in s_1$.

(ii) b) Case $a^q \in cs$. As above we deal with the 3 cases $\alpha) 1/b \notin c, \beta) 1/b \in c_0$ and $\gamma) 1/b \in c \setminus c_0$. Case $\alpha)$. We have $x \in S_p^c$ implies $(\ell^p_a)_\Delta \subset s_b^{(c)}$ and $D_{1/b} \Sigma D_a \in (\ell^p, c)$. From the characterization of $(\ell^p, c)$ we deduce $1/b \in c$. We conclude that if $1/b \notin c$, then $S_p^c = \emptyset$. Case $\beta)$. We have $x \in S_p^c$ implies

$$x \in s_b^{(c)}$$

and

$$s_b^{(c)} \subset (\ell^p_a)_\Delta + s_x^{(c)}.$$ Again using similar arguments that as those in Lemma 7, we easily see that since $b \in s_b^{(c)}$ there are $\lambda \in \ell^p$ and $\varphi \in c$ such that

$$\frac{b_n}{x_n} \left(1 - \frac{1}{b_n} \sum_{k=1}^n a_k \lambda_k\right) = \varphi_n \text{ for all } n.$$ From the characterization of $(\ell^p, c_0)$, (cf. [16, Theorem 1.37, pp. 160-161]) we have $D_{1/b} \Sigma D_a \in (\ell^p, c_0)$ since $1/b \in c_0$ and $a^q \in cs$ together imply $(b_n^{-q} (a_1^q + \ldots + a_n^q))_n \in s_1$. We deduce

$$(D_{1/b} \Sigma D_a)_n \lambda = \frac{1}{b_n} \sum_{k=1}^n a_k \lambda_k \rightarrow 0 \quad (n \rightarrow \infty),$$ and $b/x \in c$. Using the condition in (15) we conclude $x \in S_p^c$ implies $s_x^{(c)} = s_b^{(c)}$. Conversely, assume $s_x^{(c)} = s_b^{(c)}$. Since $1/b \in c_0$ and $a^q \in cs$ together imply $D_{1/b} \Sigma D_a \in (\ell^p, c)$, (cf. [16, Theorem 1.37]), we successively obtain $(\ell^p_a)_\Delta \subset s_b^{(c)}$, $(\ell^p_a)_\Delta + s_x^{(c)} = (\ell^p_a)_\Delta + s_b^{(c)} = s_b^{(c)}$ and $x \in S_p^c$. We conclude $S_p^c = \text{cl}^c (b)$.

Case $\gamma)$. Here we have $s_b^{(c)} = c$ and we are led to study the (SSE)

$$(\ell^p_a)_\Delta + s_x^{(c)} = c.$$ We have $x \in S_p^c$ implies $s_x^{(c)} \subset c$, that is, $x_n \rightarrow l$ ($n \rightarrow \infty$). Then by Lemma 7 (ii) with $E = \ell^p$ and $F = c$, the condition $x \in S_p^c$ implies $x \in I \ell^p$ and $x \in s_1$.

This means $l > 0$ and $S_p^c = \text{cl}^c (e)$. This completes the proof.

From Theorem 1 we immediately obtain the following.
Corollary 1. Let $b \in U^+$ and let $S$ be the set of all positive sequences $x$ that satisfy the (SSE) $(c_0) \Delta + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,

(i) $S \neq \emptyset$,

(ii) $S = \text{cl}^{c}(b)$,

(iii) $1/b \in s_{(1/n)^n}$.

We also obtain the following corollary, where $bv_p = \ell_p^\Delta$ is the set of all sequences of $p$-bounded variation.

Corollary 2. Let $b \in U^+$ and $p > 1$, and denote by $S_p$ the set of all positive sequences $x$ that satisfy the (SSE) $bv_p + s_x^{(c)} = s_b^{(c)}$. Then the next statements are equivalent, where,

(i) $S_p \neq \emptyset$,

(ii) $S_p = \text{cl}^{c}(b)$,

(iii) $1/b \in s_{(1/n^q)}$ with $q = p/(p-1)$.

6.2. The equation $s_x^{(c)} = s_b^{(c)}$ and the perturbed equation

\[
(s_0^a) \Delta + s_x^{(c)} = s_b^{(c)}
\]

In view of perturbed equations we can state the following. Let $b$ be a positive sequence. The equation

\[
s_x^{(c)} = s_b^{(c)}
\]

is equivalent to $x_n/b_n \to l \ (n \to \infty)$ for some $l > 0$. Then the (SSE)

\[
(s_0^a) \Delta + s_x^{(c)} = s_b^{(c)}
\]

can be considered as a perturbed equation of (18), and the question is: what are the conditions on $a$ for which the perturbed equation and the (SSE) defined by (18) have the same solutions. As a direct consequence of Theorem 1 we obtain the next corollary, where $c\overline{s}$ is the complement of $cs$.

Corollary 3. Let $a, b \in U^+$. Then we have

(i) if $1/b \in c$, then the equations in (18) and (19) are equivalent if and only if $a \in cs \cup (c\overline{s} \cap (s_b)_{\Sigma})$.

(ii) If $1/b \notin c$, the perturbed equation in (19) has no solutions.

Proof. (i) is an immediate consequence of Theorem 1. (ii) Let $a \notin cs$. The condition $\sigma \in s_1$ should imply $1/b_n \leq K (\sum_{k=1}^{n} a_k)^{-1}$ for all $n$ and for some $K > 0$ and $1/b \in c_0$, which is contradictory. So the perturbed equation in (19) has no solutions. The case $a \in cs$ is a direct consequence of Theorem 1 (i) b).
Remark 1. We may state a similar result for the perturbed equation 
\((\ell_p^a)_{\Delta} + s_x^{(c)} = s_b^{(c)}\).

6.3. Cases when \(b\), or \(b^q\) is in \(\widehat{C_1}\).

Now we state the next elementary results, where \(\widehat{C_1}\) is the set of all positive sequences \(x\) that satisfy \((x_n^{-1} \sum_{k=1}^n x_k)_n \in \ell_\infty\), (cf. [1]).

Corollary 4. Let \(a, b \in U^+\). Then we have

(i) Let \(b \in \widehat{C_1}\). Then the set \(S_0^c(\Sigma)\) of all positive \(x \in U^+\) such that \((\ell_p^a)_{\Delta} + s_x^{(c)} = s_b^{(c)}\) is determined in the following way.

a) Let \(a \not\in cs\). Then we have

\[
S_0^c = \begin{cases} 
cl^c(b), & \text{if } a/b \in s_{1}, \\
\emptyset, & \text{if } a/b \not\in s_{1}. 
\end{cases}
\] (20)

b) Let \(a \in cs\). Then we have \(S_0^c = cl^c(b)\).

(ii) Let \(p > 1\) and let \(b^q \in \widehat{C_1}\) with \(q = p/(p-1)\). Then the set \(S_p^c\) of all \(x \in U^+\) such that \((\ell_p^a)_{\Delta} + s_x^{(c)} = s_b^{(c)}\) is determined in the following way.

a) Let \(a^q \not\in cs\). Then \(S_p^c = S_0^c\) defined by (20).

b) Let \(a^q \in cs\). Then \(S_p^c = S_0^c = cl^c(b)\).

Proof. (i) a) We have \(\sigma \in s_{1}\) if and only if \(a \in s_b(\Sigma)\). But by [1, Theorem 2.6, p. 1789] we have \(b \in \widehat{C_1}\) if and only if \(s_b(\Delta) = s_b\). This implies that \(\Delta \in (s_b, s_b)\) is bijective and so is for \(\Sigma = \Delta^{-1}\). So we have \(s_b(\Sigma) = s_b\). We have \(\sigma \in s_{1}\) if and only if \(a/b \in s_{1}\), and we conclude by Theorem 1. This completes the proof of (i) a).

(i) b) comes from that fact that \(b \in \widehat{C_1}\) implies \(1/b \in c_0\), (see [1, Proposition 2.1, p. 1786]).

(ii) a) Here we have

\[
\left(\frac{a_1^q + \ldots + a_n^q}{b_n^q}\right)_n \in s_{1}\text{ if and only if } a^q \in s_{b^q}(\Sigma),
\]

and as we have just seen we have \(s_{b^q}(\Sigma) = s_{b^q}\) since \(b^q \in \widehat{C_1}\). So we obtain (ii) a). (ii) b) we have \(b^q \in \widehat{C_1}\) implies that there are \(C > 0\) and \(\gamma > 1\) such that \(b_n^q \geq C\gamma^n\), for all \(n\), (cf. [1, Proposition 2.1, p. 1786]). So we have \(b_n \geq C^{1/q}\gamma^n/q\) for all \(n\), and \(1/b \in c_0\). We conclude by Theorem 1. This completes the proof. □

Remark 2. Notice that for \(b \in \widehat{C_1}\) we have \(S_0^c \neq \emptyset\) if and only if \(a \in (cs \cup (\overline{cs} \cap s_b)) \cap U^+\).
Example 1. Consider the (SSE) with operator defined by

\[
\left( s_{(n^{-\alpha})}^0 \right)_\Delta + s_x^{(c)} = s_b^{(c)},
\]

with \( 0 < \alpha \leq 1 \) and \( b \in \hat{C}_1 \). We have \( a/b = (n^{-\alpha}/b_n)_n \). By [1, Proposition 2.1, p 1786], \( b \in \hat{C}_1 \) implies that there are \( K > 0 \) and \( \gamma > 1 \) such that \( b_n \geq K\gamma^n \) for all \( n \). This implies \( a/b \in c_0 \). We may apply Corollary 4 and conclude that the solutions of the (SSE) in (21) satisfy the condition \( x_n \sim Cb_n \) \((n \to \infty)\) for some \( C > 0 \).

Example 2. Let \( b^q \in \hat{C}_1 \). It can easily be shown that the solutions of the (SSE) \( \left( \ell^p_{(n^\alpha)} \right)_\Delta + s_x^{(c)} = s_b^{(c)} \) are defined by \( x_n \sim Cb_n \) \((n \to \infty)\) for some \( C > 0 \) and for all reals \( \alpha \).

Remark 3. Notice that if \( a \in \hat{C}_1 \), the set \( S_0^c = S(s_0^a, c) \) is determined by Corollary 4 \((i)\). Indeed, \( a \in \hat{C}_1 \) implies \( (s_0^a)_\Delta = s_0^a \), (cf. [1, Theorem 2.6, p. 1789]) and we conclude from the solvability of the (SSE) \( s_a^0 + s_x^{(c)} = s_b^{(c)} \), (cf. [11, Theorem 4.4, p. 7]).

Remark 4. If \( \lim_{n \to \infty} (a_{n-1}/a_n) < 1 \), then \( \left( \ell^p_a \right)_\Delta = \ell^p_a \), (cf. [2, Theorem 6.5 p. 3200]). So we have \( S_b^c = cl^c(b) \) if \( a/b \in s_1 \), and \( S_b^c = \emptyset \) if \( a/b \notin s_1 \).

7. Applications to particular (SSE) where \( a \) and \( b \) are classical sequences

7.1. On the (SSE) \( \left( s_R^0 \right)_\Delta + s_x^{(c)} = s_R^{(c)} \)

We obtain the next corollary whose the proof is elementary and is left to the reader.

Corollary 5. Let \( R, \overline{R} > 0 \), and denote by \( S_{R, \overline{R}} \) the set of all positive sequences \( x \) that satify the (SSE) \( \left( s_R^0 \right)_\Delta + s_x^{(c)} = s_R^{(c)} \). Then we obtain

(i) Case \( R < 1 \). We have

\[
S_{R, \overline{R}} = \begin{cases} 
cl^c(\overline{R}), & \text{if } \overline{R} \geq 1, \\
\emptyset, & \text{if } \overline{R} < 1.
\end{cases}
\]

(ii) Case \( R = 1 \). We have

\[
S_{R, \overline{R}} = \begin{cases} 
cl^c(\overline{R}), & \text{if } \overline{R} > 1, \\
\emptyset, & \text{if } \overline{R} \leq 1.
\end{cases}
\]
(iii) Case $R > 1$. We have
\[
S_{R, \overline{R}} = \begin{cases} 
  \text{cl}^c(\overline{R}), & \text{if } R \leq \overline{R}, \\
  \emptyset, & \text{if } R > \overline{R}.
\end{cases}
\]

As a direct consequence of the preceding we can state the next remark.

**Remark 5.** Let $R, \overline{R} > 0$. We have $S_{R, \overline{R}} \neq \emptyset$ if and only if $R = 1 < \overline{R}$, or $1 < R \leq \overline{R}$, or $R < 1 \leq \overline{R}$. For instance the set of all positive sequences that satisfy the (SSE) $(s^0_{\overline{R}})_\Delta + s^{(c)}_x = s^{(c)}_2$ is non empty if and only if $R \leq 2$.

7.2. On the (SSE) $(s^0_{1/r})_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(1/n^\alpha)_n}$ and
\[
(s^0_{(n^{-\alpha})_n})_\Delta + s^{(c)}_{1/x} = s^{(c)}_{1/r}.
\]

7.2.1. The (SSE) $(s^0_{1/r})_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(1/n^\alpha)_n}$

Now we consider the next statement: the condition $n^\alpha y_n \to l_1$ ($n \to \infty$) holds if and only if there are two sequences $u, v$, with $y = u + v$ such that
\[
r^n (u_n - u_{n-1}) \to 0 \text{ and } x_n v_n \to l_2 \text{ ($n \to \infty$)}
\]
for some scalars $l_1, l_2$ and for all $y \in \omega$. The set of all $x$ that satisfy the previous statement is equivalent to the (SSE)

\[
(22) \quad (s^0_{1/r})_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(1/n^\alpha)_n}.
\]

We obtain the following.

**Corollary 6.** Let $r > 0$ and $\alpha$ be a real and let $\overline{S}_{r, \alpha}$ be the set of all positive sequences $x$ that satisfy the (SSE) defined by (22). Then we obtain

(i) if $r < 1$, then $\overline{S}_{r, \alpha} = \emptyset$.

(ii) If $r = 1$, then we have
\[
\overline{S}_{r, \alpha} = \begin{cases} 
  \text{cl}^c((n^\alpha)_n), & \text{if } \alpha \leq -1, \\
  \emptyset, & \text{if } \alpha > -1.
\end{cases}
\]

(iii) If $r > 1$, then we have
\[
\overline{S}_{r, \alpha} = \begin{cases} 
  \text{cl}^c((n^\alpha)_n), & \text{if } \alpha \leq 0, \\
  \emptyset, & \text{if } \alpha > 0.
\end{cases}
\]
Proof. Notice that \( r < 1 \) implies \( a = (r^{-n})_n \notin cs \). So the statement in (i) comes from the equivalence \( \sigma_n \sim (1 - r)^{-1} n^{\alpha} r^{-n} \) (\( n \to \infty \)) and \( \sigma \notin s_1 \), for \( r < 1 \). Let \( r = 1 \). Then we have \( \sigma_n \sim n^{\alpha+1} \) (\( n \to \infty \)) and \( \sigma \in s_1 \) if and only if \( \alpha \leq -1 \), and we conclude by conclude by Theorem 1. This shows ii). Finally for \( r > 1 \), we have \( a \in cs \) and \( 1/b = (n^\alpha)_n \in c \) if and only if \( \alpha \leq 0 \), and we conclude by Theorem 1. This completes the proof. ■

We immediately deduce the next remark.

Remark 6. We have \( S_{r, \alpha} \neq \emptyset \) if and only if \( r = 1 \leq \alpha \), or \( r > 1 \) and \( \alpha \leq 0 \). We also have \( S_{r, 0} \neq \emptyset \) if and only if \( r > 1 \).

Example 3. Consider the statement: \( y_n/n \to l_1 \) (\( n \to \infty \)) holds if and only if there are two sequences \( u, v \), with \( y = u + v \) such that \( u_n - u_{n-1} \to 0 \) and \( x_n v_n \to l_2 \) (\( n \to \infty \)) for some scalars \( l_1, l_2 \) and for all \( y \in \omega \). This statement holds if and only if \( x \in S_{1, -1} \), that is, \( x_n/n \to L \) (\( n \to \infty \)) with \( L > 0 \).

7.2.2. On the \((\text{SSE})\) \( \left(s_{(1/n^\alpha)}^0\right)_n \Delta + s_{1/x}^c = s_{1/r}^c \)

As an application of Theorem 1 the following can easily be shown.

Corollary 7. Let \( r > 0 \), \( \alpha \) be a real and \( \overline{S}_{\alpha, r} \) be the set of all \( x \in U^+ \) such that \( \left(s_{(1/n^\alpha)}^0\right)_n \Delta + s_{1/x}^c = s_{1/r}^c \). The next statements are equivalent.

(\text{i}) \( \overline{S}_{\alpha, r} \neq \emptyset \), 
(\text{ii}) \( \overline{S}_{\alpha, r} = clc ((r^n)_n) \), 
(\text{iii}) \( r \leq 1 < \alpha \), or \( \alpha \leq 1 \) and \( r < 1 \).

Proof. This result is a direct consequence of the equivalences

\[
f_n = \sum_{k=1}^n k^{-\alpha} \sim \frac{n^{1-\alpha}}{1 - \alpha} \quad (n \to \infty)
\]

if \( \alpha \neq 1 \); and

\[
f_n \sim \ln n \quad (n \to \infty)
\]

if \( \alpha = 1 \).

Then if \( \alpha \neq 1 \), we have \( (r^n n^{1-\alpha})_n \in \ell_\infty \) if and only if \( r \leq 1 < \alpha \), or \( \alpha \) and \( r < 1 \); and if \( \alpha = 1 \) we have \( (r^n \ln n)_n \in \ell_\infty \) if and only if \( r < 1 \). This concludes the proof. ■

Example 4. For \( r = 1/2 \) we have \( \overline{S}_{\alpha, 1/2} = clc ((2^{-n})_n) \) for all reals \( \alpha \).

7.3. On the \((\text{SSE})\) \( \left(s_{(1/n^\alpha)}^0\right)_n \Delta + s_{1/x}^c = s_{(1/n^\beta)}^c \)

Now let \( S_{\alpha, \beta} \) for all reals \( \alpha \) and \( \beta \), be the set of all positive sequences \( x = (x_n)_n \) that satisfy the following statement. For every \( y \) the condition
$n^\beta y_n \to l_1$ ($n \to \infty$) holds if and only if there are two sequences $u, v$, with $y = u + v$ such that $n^\alpha (u_n - u_{n-1}) \to 0$ and $x_n v_n \to l_2$ ($n \to \infty$) for some scalars $l_1, l_2$. This statement leads to the solvability of the (SSE) defined by $\left(s_0^{\alpha n} \right)_\Delta + s^{(c)}_{1/x} = s^{(c)}_{\alpha (1/n^\beta)}$. We obtain the next result which can be obtained by similar arguments as those used above.

**Corollary 8.** Let $\alpha, \beta$ be reals. Then

(i) if $\alpha < 1$, then we have

$$S_{\alpha, \beta} = \begin{cases} \text{cl}^c (\langle n^\beta \rangle_n), & \text{if } \beta \leq \alpha - 1, \\ \emptyset, & \text{if } \beta > \alpha - 1. \end{cases}$$

(ii) If $\alpha = 1$, then we have

$$S_{\alpha, \beta} = \begin{cases} \text{cl}^c (\langle n^\beta \rangle_n), & \text{if } \beta < 0, \\ \emptyset, & \text{if } \beta \geq 0. \end{cases}$$

(iii) If $\alpha > 1$, then we have

$$S_{\alpha, \beta} = \begin{cases} \text{cl}^c (\langle n^\beta \rangle_n), & \text{if } \beta \leq 0, \\ \emptyset, & \text{if } \beta > 0. \end{cases}$$

**Corollary 9.** $S_{\alpha, \beta} \neq \emptyset$ if and only if $\beta \leq \alpha - 1 < 0$, or $\alpha = 1$ and $\beta < 0$, or $\alpha > 1$ and $\beta \leq 0$.

**Example 5.** As a direct consequence of the preceding, notice that the (SSE) $\left(s_0^{\alpha n} \right)_\Delta + s^{(c)}_{1/x} = c$ is equivalent to $x_n \to L$ ($n \to \infty$) with $L > 0$, for all $\alpha > 1$.

**7.4. On the (SSE) $\left(\ell^p_{(-\alpha)} \right)_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(-\beta)}$**

In the next corollary we deal with the statement for reals $\alpha$ and $\beta$ and $p > 1$: the condition $n^\beta y_n \to l_1$ holds if and only if there are $u, v \in \omega$ with $y = u + v$ such that $\sum_{k=1}^{\infty} (k^\alpha |u_k - u_{k-1}|)^p < \infty$ and $x_n v_n \to l_1$ ($n \to \infty$) for all $y \in \omega$, and for some scalars $l_1, l_2$. This is equivalent to the (SSE)

$$(23) \quad \left(\ell^p_{(-\alpha)} \right)_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(-\beta)}.$$  

We obtain the next result.

**Corollary 10.** Let $\alpha$ and $\beta$ be reals and let $S^c_p$ be the set of all the solutions of the (SSE) determined by (23). Then we have
(i) If $\alpha q \geq 1$, then we have

$$S^c_p = \begin{cases} cl^c((n^\beta)_n), & \text{if } \beta < 0, \\ \emptyset, & \text{if } \beta \geq 0. \end{cases}$$

(ii) If $\alpha q < 1$, then we have

$$S^c_p = \begin{cases} cl^c((n^\beta)_n), & \text{if } \alpha - \beta \geq \frac{1}{q}, \\ \emptyset, & \text{if } \alpha - \beta < \frac{1}{q}. \end{cases}$$

**Proof.** The proof comes from the fact that

$$\sigma_n \sim \frac{n^{(\beta-\alpha)q+1}}{1-\alpha q} \quad (n \to \infty)$$

if $\alpha q \neq 1$; and $\sigma_n \sim n^{\beta q}\ln n \quad (n \to \infty)$ if $\alpha q = 1$. Then it can easily be seen that $\sigma \in \ell_\infty$ if and only if $\alpha - \beta \geq 1/q$ for $\alpha q < 1$, or $\beta < 0$ for $\alpha q \geq 1$. We conclude by Theorem 1.

We deal for reals $\beta$ with the statement: $n^\beta y_n \to l_1$ if and only if $y = u + v$ with

$$\sum_{k=1}^{\infty} \left( \frac{|u_k - u_{k-1}|}{k^\beta} \right)^2 < \infty$$

and for some scalars $l_1, l_2$. This statement is equivalent to the (SSE) defined by

$$\left( \ell_{(n)_n}^2 \right)_\Delta + s^{(c)}_{1/x} = s^{(c)}_{(n-\beta)_n}$$

and this (SSE) has solutions if and only if $\beta \leq -3/2$.

**Example 6.** Notice that the set of all the solutions of the (SSE) defined by

$$\left( \ell_{(1/\sqrt{n})_n}^2 \right)_\Delta + s^{(c)}_{x} = s^{(c)}_{(\ln n)_n}$$

are determined by $\lim_{n \to \infty} (x_n / \ln n) > 0$. This result comes from the equivalence

$$\sum_{k=1}^{n} \left( \frac{1/\sqrt{k}}{k} \right)^2 \sim \ln n \quad (n \to \infty).$$

**8. Solvability of the (SSE) of the form** $(E_a)_\Delta + s^0_x = s^0_b$

In this section we solve the (SSE) $(E_a)_\Delta + s^0_x = s^0_b$ where $E = c$, or $\ell_\infty$. For $E = c$, the solvability of the previous (SSE) consists in determining the set of all positive sequences $x = (x_n)_n$ that satisfy the next statement. For every $y$ the condition $y_n/b_n \to 0 \quad (n \to \infty)$ holds if and only if there are two sequences $u, v$, with $y = u + v$ such that $(u_n - u_{n-1})/a_n \to l$ and $v_n/x_n \to 0 \quad (n \to \infty)$ for some scalar $l$. Here also we may consider the (SSE)
\[
(s_{a}^{(c)})_{\Delta} + s_{x}^{0} = s_{b}^{0}
\]
as a perturbed equation of the equation \( s_{x}^{0} = s_{b}^{0} \), which is equivalent to \( K_{1} \leq x_{n}/b_{n} \leq K_{2} \) for all \( n \) and for some \( K_{1}, K_{2} > 0 \). We obtain the equivalence of these two equations under some conditions on \( a \) and \( b \).

8.1. Solvability of the (SSE) \((E_{a})_{\Delta} + s_{x}^{0} = s_{b}^{0}\) where \( E = c \),
or \( \ell_{\infty} \) in the general case

To prove the next result we need a lemma.

**Lemma 8.** Let \( b \in U^{+} \) and let \( T \) be a triangle. Then we have

\[
s_{b}^{0} \subset Ts_{1} \text{ if and only if } s_{b} \subset Ts_{1}.
\]

**Proof.** We have \( s_{b}^{0} \subset Ts_{1} \) if and only if

\[
T^{-1}D_{b} \in (c_{0}, s_{1}).
\]

Since \((c_{0}, s_{1}) = (s_{1}, s_{1})\) the condition in (24) is equivalent to \( s_{b} \subset Ts_{1} \). This concludes the proof. □

**Theorem 2.** The set \( S_{E}^{0} \) of all the solutions of the (SSE) \((E_{a})_{\Delta} + s_{x}^{0} = s_{b}^{0}\) where \( E = c \), or \( \ell_{\infty} \) is determined by

\[
S_{E}^{0} = \begin{cases} 
\text{cl}^{\infty}(b), & \text{if } \sigma \in c_{0}, \\
\emptyset, & \text{if } \sigma \notin c_{0}.
\end{cases}
\]

**Proof.** Let \( x \in S_{E}^{0} \). Then we have the inclusion \((E_{a})_{\Delta} + s_{x}^{0} \subset s_{b}^{0}\). This implies \((E_{a})_{\Delta} \subset s_{b}^{0}\) and \( D_{1/b}\Sigma D_{a} \in (E, c_{0}) \). This implies

\[
D_{1/b}\Sigma D_{a} \in (c, c_{0})
\]
since \( E \supset c \) and

\[
\sigma_{n} \to 0 \ (n \to \infty).
\]

Now we have \( s_{x}^{0} \subset s_{b}^{0} \) and

\[
x \in s_{b}.
\]

Then we consider the (SSIE) defined by

\[
(27) \quad s_{b}^{0} \subset (E_{a})_{\Delta} + s_{x}^{0}.
\]
The (SSIE) in (27) with $E \subset s_1$ implies $s_b^0 \subset Ts_1$ with $T = \Sigma D_a + D_x$. So the inclusion in (27) implies

$$s_b \subset (s_a)_\Delta + s_x$$

by Lemma 8. Since $(s_a)_\Delta = \Sigma s_a$ there are $h, k \in s_1$ such that

$$\frac{b_n}{x_n} \left( 1 - \frac{1}{b_n} \sum_{k=1}^n a_k h_k \right) = k_n \text{ for all } n.$$ 

We have

$$\left| \frac{1}{b_n} \sum_{k=1}^n a_k h_k \right| \leq K\sigma_n \text{ for all } n \text{ and for some } K > 0,$$

and from the condition in (25) we deduce $b/x \in s_1$. Using the condition in (26) we conclude $x \in S^0_E$ implies $x \in c_l^\infty (b)$. Conversely, assume $x \in c_l^\infty (b)$ and (25) holds. Since $1/b \in c_0^0$, we have $(E_a)_\Delta \subset s_b^0$ for $E = c_0$, or $s_1$, we obtain

$$(E_a)_\Delta + s_b^0 = s_b^0,$$

and $x \in S^0_E$. We conclude $S^0_E = c_l^\infty (b)$. 

For $a = e$ we easily obtain the next result.

**Corollary 11.** (i) The set $S(E_\Delta, c_0)$ of all the solutions of the (SSE) $E_\Delta + s^0_x = s_b^0$ where $E = c$, or $\ell_\infty$ is determined by

$$S(E_\Delta, c_0) = \begin{cases} 
cl^\infty (b) & \text{if } (n/b_n)_n \in c_0, \\
\emptyset & \text{if } (n/b_n)_n \notin c_0.
\end{cases}$$

**Example 7.** The equation $E_\Delta + s^0_x = s_0^{(n^\alpha)}$ where $E = c$, or $\ell_\infty$, has solutions if and only if $\alpha > 1$. So the equation $E_\Delta + s^0_x = c_0$ has no solution, and the solutions of the equation $E_\Delta + s^0_x = s_0^{(n^2)}$ are determined by $K_1 n^2 \leq x_n \leq K_2 n^2$ for all $n$ and for some $K_1, K_2 > 0$.

**8.2. Applications to the solvability of (SSE) of the form**

$$(E_a)_\Delta + s^0_x = s_b^0 \text{ for particular sequences } a \text{ and } b$$

Consider the (SSE) determined by

$$(28) \quad \left( s^{(c)}_{(n^{-\alpha})n} \right)_\Delta + s^0_{1/x} = s^0_{(n^{-\beta})n},$$

$$(29) \quad \left( s^{(c)}_R \right)_\Delta + s^0_x = s^0_{1/R},$$

$$(30) \quad \left( s^{(c)}_{(n^{-\alpha})n} \right)_\Delta + s^0_{1/x} = s^0_{1/R},$$

$$(31) \quad \left( s^{(c)}_{1/R} \right)_\Delta + s^0_{1/x} = s^0_{(n^{-\beta})},$$
with reals $\alpha$ and $\beta$ and $R, \overline{R} > 0$. We obtain the following.

**Proposition 1.** (i) The (SSE) defined in (28) has solutions if and only if $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.

(ii) The (SSE) defined in (29) has solutions if and only if $R \leq 1 < \overline{R}$, or $1 < R < \overline{R}$.

(iii) The (SSE) defined in (30) has solutions if and only if $R < 1 < \alpha$, or $R < \alpha = 1$, or $\alpha$ and $R < 1$.

(iv) The (SSE) defined in (31) has solutions if and only if $R = 1$ and $\beta < -1$, or $R > 1$ and $\beta < 0$.

**Proof.** (i) It can easily be seen that $\sigma_n \sim n^{\beta-\alpha+1}/(1 - \alpha) (n \to \infty)$ for $\alpha < 1$; $\sigma_n \sim n^\beta \ln n (n \to \infty)$ for $\alpha = 1$; and $\sigma_n \sim Kn^\beta (n \to \infty)$ for $\alpha > 1$. We conclude the (SSE) defined by (28) has solutions if and only if $\sigma_n = o(1) (n \to \infty)$, that is, for $\beta < \alpha - 1 < 0$, or $\alpha \geq 1$ and $\beta < 0$.

(ii) We have

$$\sigma_n \sim \frac{1}{R - 1} \frac{R^{n+1}}{R^n} (n \to \infty) \text{ for } R > 1;$$

$$\sigma_n \sim \frac{R}{1 - R} \frac{1}{R^n} (n \to \infty) \text{ for } R < 1;$$

and

$$\sigma_n \sim \frac{n}{R^n} (n \to \infty) \text{ for } R = 1;$$

and we conclude as above.

(iii) This result is a direct consequence of the following. We have $\sigma_n \sim KR^n (n \to \infty)$ for $\alpha > 1$ and for some $K > 0$; $\sigma_n \sim n^n \ln n (n \to \infty)$ for $\alpha = 1$; and $\sigma_n \sim n^{1-\alpha} R^n/(1 - \alpha) (n \to \infty)$ for $\alpha < 1$. We conclude by Theorem 2.

(iv) Here we have $\sigma_n \sim n^\beta/(R - 1) (n \to \infty)$ for $R > 1$; $\sigma_n \sim n^{\beta+1} (n \to \infty)$ for $R = 1$; and $\sigma_n \sim n^\beta R^{-n}/(1 - R) (n \to \infty)$ for $R < 1$ and we conclude by Theorem 2.

**Example 8.** The (SSE) $\left( s^{(c)}_{1/2} \right)_\Delta + s^0_x = s^0_{R}$ has solutions if and only if $\overline{R} > 1$.

**Example 9.** Let $\tau, \tau'$ be reals. Then the system of (SSE) defined by

$$\begin{cases} c_\Delta + s^0_x = s^0_{(\tau)}_n, \\ \left( s^{(c)}_{1/2} \right)_\Delta + s^0_x = s^0_{(\tau')}_n, \end{cases}$$

where $x$ is the unknown, has solutions if and only if $\tau = \tau' > 1$. Then $x$ is a solution of the system if and only if $x_n \sim C n^\tau (n \to \infty)$ for some $C > 0.$
This is a direct consequence of Proposition 1 (iv) and of the elementary fact that \(s_{(n\tau)k} = s_{(n\tau')k}\) if and only if \(\tau = \tau'\).

Example 10. Let \(S_0^c\) be the set of all positive sequences that satisfy the following statement. For every \(y\) the condition \(y_n/n \to 0\) \((n \to \infty)\) holds if and only if there are two sequences \(u, v\), with \(y = u + v\) such that \(\sqrt{n} (u_n - u_{n-1}) \to L\) and \(x_n v_n \to 0\) \((n \to \infty)\) for some scalar \(L\). By Proposition 1 (i), we have \(x \in S_0^c\) if and only if \(K_1/n \leq x_n \leq K_2/n\) for all \(n\) and for some \(K_1, K_2 > 0\).

References


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