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SOME ADVANCES IN THE THEORY OF QUASI-PSEUDOMETRIC TYPE SPACES

Abstract. In this paper, we extend most of the results proved in [4]. In particular, we give some topological properties of the quasi-pseudometric type spaces. Moreover, some fixed point and common fixed point theorems are obtained in the setting of quasi-pseudometric spaces, introduced some months ago by Kazeem et al in [4].

Key words: quasi-pseudometric type spaces, fixed point, left K-completeness.

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Symmetric spaces were introduced in 1931 by Wilson [6], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained. In the same dynamics, cone metric spaces were introduced by Huang [3] and many fixed point results concerning mappings in these spaces have also been established. In [5], M. A. Khamsi connected this concept with a generalised form of metric that he named metric type. Namely, he observed that if \( d(x, y) \) is a cone metric, then \( D(x, y) = \|d(x, y)\| \) is symmetric with some special properties, particularly in the case when the underlying cone is normal. Recently in [4], Kazeem et al. discussed the newly introduced notion of quasi-pseudometric type spaces as a logical equivalent to metric type spaces when the initial distance-like function is not symmetric. Some fixed point results of mappings on such spaces were discussed as well in [4]. It is the aim of this article to continue the study of quasi-pseudometric spaces by proving several other fixed point and common fixed point results, hence extending the fixed point results of [4] to a class of mappings satisfying more general contractive conditions.

In this section, we recall briefly some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below. For recent results and detailed explanations for the concepts in the theory of asymmetric spaces, the reader is referred to [2, 4, 7, 8].
Definition 1. Let $E$ be a real Banach space with norm $\|\cdot\|$ and $P$ be a subset of $E$. Then $P$ is called a cone if and only if

(a) $P$ is closed, nonempty and $P \neq \{\theta\}$, where $\theta$ is the zero vector in $E$;
(b) for any $a, b \geq 0$, and $x, y \in P$, we have $ax + by \in P$;
(c) for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone $P$ in a Banach space $E$, we define on $E$ a partial order $\preceq$ with respect to $P$ by

$$x \preceq y \iff y - x \in P.$$ 

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}(P)$ (where $\text{Int}(P)$ designates the interior of $P$).

The cone $P$ is called normal if there is a number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \implies \|x\| \leq C\|y\|.$$ 

The least positive number satisfying this inequality is called the normal constant of $P$. Therefore, we shall then say that $P$ is a $K$-normal cone to indicate the fact that the normal constant is $K$.

Definition 2 (Compare [4]). Let $X$ be a nonempty set. Suppose the mapping $q : X \times X \to E$ satisfies

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, y) = \theta = q(y, x)$ if and only if $x = y$;
(q3) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$.

Then, $q$ is called a quasi-cone metric on $X$, and $(X, q)$ is called a quasi-cone metric space.

Definition 3 (Compare [4]). A sequence in a quasi-cone metric space $(X, q)$ is called

(a) $Q$-Cauchy or bi-Cauchy if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall \ n, m \geq n_0 \ q(x_n, x_m) \ll c;$$

(b) left(right) Cauchy if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall \ n, m : n_0 \leq m \leq n \ q(x_m, x_n) \ll c \ (q(x_n, x_m) \ll c \text{ resp.}).$$

Remark 1. A sequence is $Q$-Cauchy if and only if it is both left and right Cauchy.

Definition 4. (a) In a quasi-cone metric space $(X, q)$, we say that the sequence $(x_n)$ left converges to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists $N$ such that for all $n > N$, $q(x_n, x) \ll c$. 


Similarly, in a quasi-cone metric space \((X, q)\), we say that a sequence \((x_n)\) **right converges** to \(x \in X\) if for every \(c \in E\) with \(\theta \ll c\) there exists \(N\) such that for all \(n > N\), \(q(x, x_n) \ll c\).

Finally, in a quasi-cone metric space \((X, q)\), we say that the sequence \((x_n)\) converges to \(x \in X\) if for every \(c \in E\) with \(\theta \ll c\) there exists \(N\) such that for all \(n > N\), \(q(x_n, x) \ll c\) and \(q(x, x_n) \ll c\).

**Definition 5.** A quasi-cone metric space \((X, q)\) is called
(a) **left complete** (resp. right complete) if every left Cauchy (resp. right Cauchy) sequence in \(X\) left (resp. right) converges.
(b) **bicomplete** if every \(Q\)-Cauchy sequence converges.

**Remark 2.** A quasi-cone metric space \((X, q)\) is bicomplete if and only if it is left complete and right complete.

**Definition 6.** Let \((X, q)\) be a quasi-cone metric space. A function \(f : X \to X\) is said to be **Lipschitzian** if there exists some \(\kappa \in \mathbb{R}\) such that

\[q(f(x), f(y)) \leq \kappa q(x, y) \quad \forall \ x, y \in X.\]

The smallest constant which satisfies the above inequality is called the **Lipschitz constant** of \(f\) and is denoted \(\text{Lip}(f)\). In particular \(f\) is said to be **contractive** if \(\text{Lip}(f) \in [0, 1)\) and **nonexpansive** if \(\text{Lip}(f) \leq 1\).

**Definition 7** (Compare [1]). Let \(f\) and \(g\) be self maps on a set \(X\). If \(w = fx = gx\) for some \(x \in X\), then \(x\) is called a **coincidence point** of \(f\) and \(g\), and \(w\) is called the **point of coincidence** of \(f\) and \(g\).

**Definition 8.** Let \(f\) and \(g\) be self maps on a nonempty set \(X\). We say that \(f\) and \(g\) are **weakly compatible** if they commute at their coincidence point, that is there exists \(x_0 \in X\) such that \(fx_0 = gx_0\) then \(gf x_0 = fg x_0\).

We also give the following proposition that we take from [1] by omitting the proof.

**Proposition 1** (Compare [1]). Let \(f\) and \(g\) be weakly compatible self maps on a set \(X\). If \(f\) and \(g\) have a unique point of coincidence \(w = fx = gx\), then \(w\) is the unique common fixed point of \(f\) and \(g\).

We also have the following important characterization

**Lemma 1.** Let \((X, q)\) be a quasi-cone metric space, \(P\) be a \(K\)-normal cone and \((x_n)\) be a sequence in \(X\). Then \((x_n)\) is a bi-Cauchy sequence if and only if \(q(x_n, x_m) \to \theta\) as \(n, m \to \infty\).

We now connect the notion of quasi-cone metric to the one of quasi-pseudo-metric type space via the following theorem.
Theorem 1 (Compare [4] Theorem 28). Let \((X, q)\) be a quasi-cone metric space over the Banach space \(E\) with the \(K\)-normal cone \(P\). The mapping \(Q : X \times X \to [0, \infty)\) defined by \(Q(x, y) = \|q(x, y)\|\) satisfies the following properties

\((Q1)\) \(Q(x, x) = 0\) for any \(x \in X\);
\((Q2)\) \(Q(x, y) \leq K(Q(x, z_1) + Q(z_1, z_2) + \cdots + Q(z_n, y))\), for any points \(x, y, z_i \in X, i = 1, 2, \ldots, n\).

We are therefore led to the following definition.

Definition 9 ([4]). Let \(X\) be a non-empty set, and let the function \(D : X \times X \to [0, \infty)\) satisfy the following properties:

\((D1)\) \(D(x, x) = 0\) for any \(x \in X\);
\((D2)\) \(D(x, y) \leq \alpha(D(x, z_1) + D(z_1, z_2) + \cdots + D(x_n, y))\) for any points \(x, y, z_i \in X, i = 1, 2, \ldots, n\) and some constant \(\alpha > 0\).

Then \((X, D, \alpha)\) is called a quasi-pseudometric type space. Moreover, if \(D(x, y) = 0 = D(y, x) \implies x = y\), then \(D\) is said to be a \(T_0\)-quasi-pseudometric type space. The latter condition is referred to as the \(T_0\)-condition.

Remark 3.  
- Let \(D\) be a quasi-pseudometric type on \(X\), then the map \(D^{-1}\) defined by \(D^{-1}(x, y) = D(y, x)\) whenever \(x, y \in X\) is also a quasi-pseudometric type on \(X\), called the conjugate of \(D\). We shall also denote \(D^{-1}\) by \(D^t\) or \(\bar{D}\).
- It is easy to verify that the function \(D^s\) defined by \(D^s := D \vee D^{-1}\), i.e. \(D^s(x, y) = \max\{D(x, y), D(y, x)\}\) defines a metric type (see [5]) on \(X\) whenever \(D\) is a \(T_0\)-quasi-pseudometric type.
- If we substitute the property \((D1)\) by the following property

\((D3) : D(x, y) = 0 \iff x = y\),

we obtain a \(T_0\)-quasi-pseudometric type space directly. For instance, this could be done if the map \(D\) is obtained from quasi-cone metric.

Moreover, for \(\alpha = 1\), we recover the classical pseudometric, hence quasi-pseudometric type spaces generalize quasi-pseudometrics. It is worth mentioning that if \((X, D, \alpha)\) is a pseudometric type space, then for any \(\beta \geq \alpha\), \((X, D, \beta)\) is also a pseudometric type space. We give the following example to illustrate the above comment.

Example 1. Let \(X = \{a, b, c\}\) and the mapping \(D : X \times X \to [0, \infty)\) defined by \(D(a, b) = D(c, b) = 1/5, D(b, c) = D(b, a) = D(c, a) = 1/4, D(a, c) = 1/2, D(x, x) = 0\) for any \(x \in X\) and \(D(x, y) = D(y, x)\) for any \(x, y \in X\). Since

\[
\frac{1}{2} = D(a, c) > D(a, b) + D(b, c) = \frac{9}{20},
\]

then \((X, D, 1)\) is a \(T_0\)-quasi-pseudometric type space.
then we conclude that $X$ is not a quasi-pseudometric space. Nevertheless, with $\alpha = 2$, it is very easy to check that $(X, D, 2)$ is a quasi-pseudometric type space.

**Definition 10** ([4]). Let $(X, D, \alpha)$ be a quasi-pseudometric space. The convergence of a sequence $(x_n)$ to $x$ with respect to $D$, called $D$-convergence or **left-convergence** and denoted by $x_n \xrightarrow{D} x$, is defined in the following way

(1) \[ x_n \xrightarrow{D} x \iff D(x, x_n) \to 0. \]

Similarly, the convergence of a sequence $(x_n)$ to $x$ with respect to $D^{-1}$, called $D^{-1}$-convergence or **right-convergence** and denoted by $x_n \xrightarrow{D^{-1}} x$, is defined in the following way

(2) \[ x_n \xrightarrow{D^{-1}} x \iff D(x_n, x) \to 0. \]

Finally, in a quasi-pseudometric space $(X, D, \alpha)$, we shall say that a sequence $(x_n)$ **$D^{s}$-converges** to $x$ if it is both left and right convergent to $x$, and we denote it as $x_n \xrightarrow{D^{s}} x$ or $x_n \xrightarrow{D} x$ when there is no confusion. Hence

\[ x_n \xrightarrow{D^{s}} x \iff x_n \xrightarrow{D} x \text{ and } x_n \xrightarrow{D^{-1}} x. \]

**Definition 11** ([4]). A sequence $(x_n)$ in a quasi-pseudometric type space $(X, D, \alpha)$ is called

(a) **left $K$-Cauchy** with respect to $D$ if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

\[ \forall n, k : n_0 \leq k \leq n \quad D(x_k, x_n) < \epsilon; \]

(b) **right $K$-Cauchy** with respect to $D$ if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

\[ \forall n, k : n_0 \leq k \leq n \quad D(x_n, x_k) < \epsilon; \]

(c) **$D^{s}$-Cauchy** if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

\[ \forall n, k \geq n_0 \quad D(x_n, x_k) < \epsilon. \]

**Remark 4.**

• A sequence is left $K$-Cauchy with respect to $d$ if and only if it is right $K$-Cauchy with respect to $D^{-1}$.

• A sequence is $D^{s}$-Cauchy if and only if it is both left and right $K$-Cauchy.

**Definition 12** ([4]). A quasi-pseudometric space $(X, D, \alpha)$ is called **left-complete** provided that any left $K$-Cauchy sequence is $D$-convergent.
Definition 13 ([4]). A quasi-pseudometric space \((X, D, \alpha)\) is called **right-complete** provided that any right \(K\)-Cauchy sequence is \(D\)-convergent.

Definition 14 ([4]). A \(T_0\)-quasi-pseudometric space \((X, D, \alpha)\) is called **bicomplete** provided that the metric \(D^s\) on \(X\) is complete.

2. First results

In [4], Kazeem et al. proved the following:

**Theorem 2.** Let \((X, q)\) be a bicomplete quasi-cone metric space, \(P\) a \(K\)-normal cone. Suppose that a mapping \(T : X \to X\) satisfies the contractive condition

\[ q(Tx, Ty) \leq k q(x, y) \quad \text{for all } x, y \in X, \]

where \(k \in [0, 1)\). Then \(T\) has a unique fixed point. Moreover for any \(x \in X\), the orbit \(\{T^n x, n \geq 0\}\) converges to the fixed point.

We start by an application of the above theorem

**Theorem 3.** Let \((X, q)\) be a bicomplete quasi-cone metric space, \(P\) a \(K\)-normal cone. Let \(T : X \to X\) be a map such that for every \(n \in \mathbb{N}\), there is \(\lambda_n \in (0, 1)\) such that

\[ q(T^n x, T^n y) \leq \lambda_n q(x, y) \quad \text{for all } x, y \in X. \]

and let \(\lim_{n \to 0} \lambda_n = 0\). Then \(T\) has a unique fixed point \(\omega \in X\).

**Proof.** Take \(\lambda\) such that \(0 < \lambda < 1\). Since \(\lim_{n \to 0} \lambda_n = 0\), there exists \(n_0 \in \mathbb{N}\) such that \(\lambda_n < \lambda\) for each \(n \geq n_0\). Then \(q(T^n x, T^n y) \leq \lambda_n q(x, y)\) for all \(x, y \in X\) whenever \(n \geq n_0\). In other words, for any \(m \geq n_0\), \(g = T^m\) satisfies

\[ q(gx, gy) \leq k q(x, y) \quad \text{for all } x, y \in X. \]

Theorem 2 implies that \(g\) has a unique fixed point, say \(\omega\). Then \(T^m \omega = \omega\), implying that \(T^{m+1} \omega = T(T^m \omega) = T^m(T \omega) = T \omega\) and \(T \omega\) is also a fixed point of \(g = T^m\). Since the fixed point is unique, it follows that \(T \omega = \omega\) and \(\omega\) is the unique fixed point of \(T\).

We now state below a generalization of this theorem.

**Theorem 4.** Let \((X, q)\) be a bicomplete quasi-cone metric space, \(P\) a \(K\)-normal cone. Suppose that a mapping \(T : X \to X\) is such that for every \(n \in \mathbb{N}\), \(T^n\) is Lipschitzian and that \(\sum_{n=0}^{\infty} \text{Lip}(T^n) < \infty\). Then \(T\) has a unique fixed point \(x^* \in X\).
Proof. Since for any $n \in \mathbb{N}$, $T^n$ is Lipschitzian, hence there exists $k_n := \text{Lip}(T^n) \geq 0$ such that

$$q(T^n x, T^n y) \leq k_n \, q(x, y) \quad \text{for all } x, y \in X.$$  

Now let $x \in X$. For any $n, h \in \mathbb{N}$, we have

$$q(T^n x, T^{n+h} x) \leq k_n \, q(x, T^h x) \leq k_n \left[ \sum_{i=0}^{h-1} q(T^i x, T^{i+1} x) \right].$$

Hence

$$q(T^n x, T^{n+h} x) \leq k_n \left( \sum_{i=0}^{h-1} k_i \right) q(x, T^n x),$$

since

$$q(T^i, T^{i+1} x) \leq k_i \, q(x, T^n x), \quad \text{for all } i \in \mathbb{N}.$$  

Since $\sum_{n=0}^{\infty} \text{Lip}(T^n)$ is convergent, then $\lim_{n \to 0} \text{Lip}(T^n) = 0$ and therefore inequality (4) entails that

$$\|q(T^n x, T^{n+h} x)\| \leq K k_n \left( \sum_{i=0}^{h-1} k_i \right) \|q(x, T^n x)\| \to 0 \quad \text{as } n \to \infty.$$  

Similarly, one shows that

$$\|q(T^{n+h} x, T^n x)\| \leq K k_n \left( \sum_{i=0}^{h-1} k_i \right) \|q(T^n x, x)\| \to 0 \quad \text{as } n \to \infty.$$  

From relations (5) and (6), we conclude that $(T^n x)$ is a bi-Cauchy sequence. Since $(X, q)$ is bicomplete, there exists $x^* \in X$ such that $(T^n x)$ converges to $x^*$. First let us show that $x^*$ is a fixed point of $T$.

On one side, we have

$$q(T^{n-1} x, x^*) \leq q(T^{n-1} x, T^n x) + q(T^n x, x^*)$$

$$\leq k_{n-1} q(x, T x) + q(T^n x, x^*),$$

and on the other side

$$q(x^*, T^{n-1} x) \leq q(x^*, T^n x) + q(T^n x, T^{n-1} x)$$

$$\leq k_{n-1} q(T x, x) + q(x^*, T^n x),$$
From (7), we have that
\[ q(Tx^*, x^*) \leq q(Tx^*, T^nx) + q(T^n x, x^*) \]
\[ \leq k_1 q(x^*, T^{n-1} x) + q(T^n x, x^*) \to \theta \text{ as } n \to \infty, \]
i.e
\[ q(Tx^*, x^*) = \theta. \]

In the same manner, from (8), we have that
\[ q(x^*, Tx^*) = \theta. \]

Hence
\[ q(Tx^*, x^*) = \theta = q(x^*, Tx^*). \]

This implies, using property \((q2)\) that \(Tx^* = x^*\). So \(x^*\) is a fixed point of \(T\). Moreover, if \(z^*\) is a fixed point of \(T\), then for all \(n \geq 1\), we have
\[ q(x^*, z^*) = q(T^n x^*, T^n z^*) \leq k_n q(x^*, z^*), \]
and
\[ q(z^*, x^*) = q(T^n z^*, T^n x^*) \leq k_n q(z^*, x^*). \]

Since \(\lim_{n \to 0} Lip(T^n) = 0\), hence \(\|q(x^*, z^*)\| = 0 = \|q(z^*, x^*)\|\) and \(x^* = z^*\). Therefore the fixed point is unique. \(\blacksquare\)

In the next section, we give some topological properties of quasi-pseudometric type spaces. Most of them deal with sequences and follow closely the classical properties of sequences pseudometric spaces.

3. Topology on Quasi-pseudometric type spaces and fixed point results

3.1. Some topological properties. Let \((X, D, \alpha)\) be a quasi-pseudometric type space. Then for each \(x \in X\) and \(\epsilon > 0\), the set
\[ B_D(x, \epsilon) = \{y \in X : D(x, y) < \epsilon\} \]
denotes the open \(\epsilon\)-ball at \(x\) with respect to \(D\). It should be noted that the collection
\[ \{B_D(x, \epsilon) : x \in X, \epsilon > 0\} \]
yields a base for the topology \(\tau(D)\) induced by \(D\) on \(X\). In a similar manner, for each \(x \in X\) and \(\epsilon \geq 0\), we define
\[ C_D(x, \epsilon) = \{y \in X : D(x, y) \leq \epsilon\}, \]
known as the closed \(\epsilon\)-ball at \(x\) with respect to \(D\).
Also the collection
\[ \{D_{d^{-1}}(x, \epsilon) : x \in X, \epsilon > 0\} \]
yields a base for the topology \(\tau(D^{-1})\) induced by \(D^{-1}\) on \(X\). The set \(C_D(x, \epsilon)\) is \(\tau(D^{-1})\)-closed, but not \(\tau(D)\)-closed in general.

The balls with respect to \(D\) are often called \textit{forward balls} and the topology \(\tau(D)\) is called \textit{forward topology}, while the balls with respect to \(D^{-1}\) are often called \textit{backward balls} and the topology \(\tau(D^{-1})\) is called \textit{backward topology}.

The topology \(\tau(D)\) of a quasi-pseudometric type space \((X, D, \alpha)\) can be defined starting with starting from the family \(\Pi_D(x)\) of neighbourhoods of an arbitrary point \(x \in X\).

\[
V \in \Pi_D(x) \iff \exists \; \epsilon > 0 \text{ such that } B_D(x, \epsilon) \subset V \\
\iff \exists \; \epsilon' > 0 \text{ such that } C_D(x, \epsilon) \subset V.
\]

To see the equivalence in the above definition, we can take for instance \(\epsilon' = \epsilon/3\).

The following proposition contains some simple properties of convergent sequences.

**Proposition 2.** Let \((x_n)\) be a sequence in quasi-pseudometric type space \((X, D, \alpha)\).

(a) If \((x_n)\) is \(D\)-convergent to \(x\) and \(D^{-1}\)-convergent to \(y\), then \(D(x, y) = 0\).

(b) If \((x_n)\) is \(D\)-convergent to \(x\) and \(D(y, x) = 0\), then \((x_n)\) is also \(D\)-convergent to \(y\).

**Proof.**

(a) Letting \(n \to \infty\) in the inequality
\[
D(x, y) \leq \alpha[D(x, x_n) + D(x_n, y)],
\]
one obtains \(D(x, y) = 0\).

(b) The result follows from the relations
\[
D(x_n, y) \leq \alpha[D(y, x) + D(x, x_n)] = \alpha D(x, x_n) \to 0.
\]

Also, the following simple remarks concerning sequences in quasi-pseudometric type spaces are true.

**Proposition 3.** Let \((x_n)\) be as sequence in a quasi-pseudometric type space \((X, D, \alpha)\).

(a) If \((x_n)\) is left \(K\)-Cauchy and has a subsequence which is \(\tau(D)\)-convergent to \(x\), then \((x_n)\) is \(\tau(D)\)-convergent to \(x\).
(b) If \( (x_n) \) is left \( K \)-Cauchy and has a subsequence which is \( \tau(D^{-1}) \)-convergent to \( x \), then \( (x_n) \) is \( \tau(D^{-1}) \)-convergent to \( x \).

**Proof.** (a) Suppose that \( (x_n) \) is left \( K \)-Cauchy and \( (x_{n_k}) \) is a subsequence of \( (x_n) \) such that \( \lim_{k \to \infty} D(x, x_{n_k}) = 0 \). For \( \epsilon > 0 \) choose \( n_0 \) such that \( n_0 \leq m \leq n \) implies \( D(x_m, x_n) < \epsilon/\alpha \), and let \( k_0 \in \mathbb{N} \) be such that \( n_{k_0} \geq n_0 \) and \( D(x, x_{n_k}) < \epsilon/\alpha \) for all \( k \geq k_0 \). Then, for \( n \geq n_{k_0} \), \( D(x, x_n) \leq \alpha[D(x, x_{n_{k_0}}) + D(x_{n_{k_0}}, x_n)] < 2\epsilon. \)

(b) Reasoning similarly, for \( n \geq n_{k_0} \) let \( k \in \mathbb{N} \) such that \( n_k \geq n \). Then
\[
D(x_n, x) \leq \alpha[D(x_n, x_{n_k}) + D(x_{n_k}, x)] < 2\epsilon.
\]

The proof of the following proposition is trivial and shall then be omitted.

**Proposition 4.** If a sequence \( (x_n) \) in a quasi-pseudometric type space \( (X, D, \alpha) \), satisfies
\[
\sum_{n=0}^{\infty} D(x_n, x_{n+1}) < \infty,
\]
then \( (x_n) \) is left \( K \)-Cauchy.

**Definition 15.** A subset \( Y \) of a quasi-pseudometric type space \( (X, D, \alpha) \) is called precompact if for every \( \epsilon > 0 \) there exists a finite subset \( Z \) of \( Y \) such that
\[
Y \subset \bigcup \{B_D(z, \epsilon) : z \in Z\}.
\]

If for every \( \epsilon > 0 \) there exists a finite subset \( Z \) of \( X \) such that (9) holds, then the set \( Y \) is called outside precompact. One obtains the same notions if one works with closed balls \( C_D(z, \epsilon) z \in Z \).

Obviously a precompact set is outside precompact, but the converse is not true. We then have the following characterization.

**Proposition 5.** Let \( (X, D, \alpha) \) be a quasi-pseudometric type space. A subset \( Y \) of \( X \) is precompact if and only if for every \( \epsilon > 0 \) there is a finite subset \( \{x_1, x_2, \ldots, x_n\} \subset X \) such that \( Y \subset \bigcup_{i=1}^{n} B_D(x_i, \epsilon) \) and \( Y \cap B_{D^{-1}}(x_i, \epsilon) \neq \emptyset \) for all \( i = 1, 2, \ldots, n \).

**Proof.** For \( \epsilon > 0 \), let \( \{x_1, x_2, \ldots, x_n\} \subset X \) such that the conditions hold for \( \epsilon/2\alpha \). If \( y_i \in Y \cap B_{D^{-1}}(x_i, \epsilon/2\alpha), i = 1, 2, \ldots, n \), then \( Y \subset \bigcup_{i=1}^{n} B_D(x_i, \epsilon) \).

Indeed, for any \( y \in Y \) there exists \( k \in \{1, 2, \ldots, n\} \) such that \( D(x_k, y) < \epsilon/2\alpha \), implying
\[
D(y_k, y) \leq \alpha[D(y_k, x_k) + D(x_k, y)] = \alpha[D^{-1}(x_k, y_k) + D(x_k, y)] < \epsilon.
\]
3.2. Fixed point results. We start with the following lemma and repeat the proof as it is in [4].

**Lemma 2** (Compare [4] Lemma 38). Let \((y_n)\) be a sequence in a quasi-pseudometric type space \((X, D, \alpha)\) such that

\[
D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n)
\]

for some \(\lambda > 0\) with \(\lambda < \min\{1, 1/\alpha\}\). Then \((y_n)\) is left \(K\)-Cauchy.

**Proof.** Let \(m < n \in \mathbb{N}\). From the condition \((D2)\) in the definition of a quasi-pseudometric type, we can write:

\[
D(y_m, y_n) \leq \alpha [D(y_m, y_m+1) + D(y_{m+1}, y_n)] \\
\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) + \alpha^2 D(y_{m+2}, y_n) \\
\vdots \\
\leq \alpha D(y_m, y_{m+1}) + \alpha^2 D(y_{m+1}, y_{m+2}) + \cdots \\
+ \alpha^{n-m-1} D(y_{n-2}, y_{n-1}) + \alpha^{n-m} D(y_{n-1}, y_n).
\]

From (10) and \(\lambda < \frac{1}{\alpha}\), the above becomes

\[
D(y_m, y_n) \leq (\alpha \lambda^m + \alpha^2 \lambda^{m+1} + \cdots + \alpha^{n-m} \lambda^{n-1}) D(y_0, y_1) \\
\leq \alpha \lambda^m (1 + \alpha \lambda + \cdots + (\alpha \lambda)^{n-1-m}) D(y_0, y_1) \\
\leq \frac{\alpha \lambda^m}{1 - \alpha \lambda} D(y_0, y_1) \to 0 \text{ as } m \to \infty.
\]

It follows that \((y_n)\) is left \(K\)-Cauchy. Similarly,

**Lemma 3.** Let \((y_n)\) be a sequence in a quasi-pseudometric type space \((X, D, \alpha)\) such that

\[
D^{-1}(y_n, y_{n+1}) \leq \lambda D^{-1}(y_{n-1}, y_n)
\]

for some \(\lambda > 0\) with \(\lambda < \min\{1, 1/\alpha\}\). Then \((y_n)\) is right \(K\)-Cauchy.

We now state our first fixed point result.

**Theorem 5.** Let \((X, D, \alpha)\) be a \(T_0\)-quasi-pseudometric type space. Suppose that \(f, g : X \to X\) are mappings such that

\[
D(fx, fy) \leq k D(gx, gy) \text{ for all } x, y \in X,
\]

where \(k < \min\{1, 1/\alpha\}\). If the range of \(g\) contains the range of \(f\) and \(g(X)\) is bicomplete, then \(f\) and \(g\) have a unique point of coincidence. Moreover if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.
Proof. Take an arbitrary \( x_0 \in X \). Choose a point \( x_1 \) in \( X \) such that \( f(x_0) = g(x_1) \). This can be done, since \( f(X) \subset g(X) \). Iterating this process, once \( x_n \) is chosen in \( X \), we can obtain \( x_{n+1} \) in \( X \) such that \( f(x_n) = g(x_{n+1}) \). Then

\[
D(gx_n, gx_{n+1}) = D(fx_{n-1}, fx_n) \leq kD(gx_{n-1}, gx_n) \\
\leq k^2D(gx_{n-2}, gx_{n-1}) \leq \ldots \leq k^nD(gx_0, gx_1).
\]

i.e.

\[
D(gx_n, gx_{n+1}) \leq k^nD(gx_0, gx_1).
\]

Similarly,

\[
D(gx_{n+1}, gx_n) \leq k^nD(gx_1, gx_0).
\]

Hence \((gx_n)\) is a bi-Cauchy sequence. Since \( g(X) \) is bicomplete, there exists \( x^* \in g(X) \) such that \((gx_n)\) \( D^* \)-converges to \( x^* \). In other words, there is a \( p^* \in X \) such that \((gx_n)\) converges to \( g(p^*) = x^* \).

Moreover

\[
D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq kD(gx_{n-1}, gp^*) \longrightarrow 0, \text{ as } n \longrightarrow,
\]

In the same way, we establish that \( D(fp^*, gx_n) \longrightarrow 0 \) as \( n \longrightarrow \infty \), to then conclude that \( gx_n \longrightarrow fp^* \). The uniqueness of the limit implies that \( fp^* = gp^* \). We finish the proof by showing that \( f \) and \( g \) have a unique point of coincidence. For this, assume \( z^* \in X \) is a point such that \( fz^* = gz^* \).

Now

\[
D(gz^*, gp^*) = D(fz^*, fp^*) \leq kD(gz^*, gp^*),
\]

which gives \( D(gz^*, gp^*) = 0 \). On the other hand, by the same reasoning, it also clear that \( D(gp^*, gz^*) = 0 \). By property the \( T_0 \)-condition, \( gz^* = gp^* \).

From Proposition 1, \( f \) and \( g \) have a unique common fixed point.

Theorem 6. Let \((X, D, \alpha)\) be a \( T_0 \)-quasi-pseudometric type space. Suppose that \( f, g : X \rightarrow X \) are mappings such that Suppose that mappings \( f, g : X \rightarrow X \) satisfy the contractive condition

\[
D(fx, fy) \leq k \left[ D(fx, gy) + D(gx, fy) \right] \text{ for all } x, y \in X,
\]

where \( k \geq 0 \) such that \( \frac{k}{1-k} < \min\{1, 1/\alpha\} \). If the range of \( g \) contains the range of \( f \) and \( g(X) \) is bicomplete, then \( f \) and \( g \) have a unique coincidence point in \( X \). Moreover if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.
Take an arbitrary $x_0 \in X$. Choose a point $x_1$ in $X$ such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once $x_n$ is chosen in $X$, we can obtain $x_{n+1}$ in $X$ such that $f(x_n) = g(x_{n+1})$. Then
\[ D(gx_n, gx_{n+1}) = D(fx_{n-1}, fx_n) \leq k[D(fx_{n-1}, gx_n) + D(gx_{n-1}, fx_n)] \]
\[ \leq kD(gx_{n-1}, gx_{n+1}) \]
\[ \leq k[D(gx_{n-1}, gx_n) + D(gx_n, gx_{n+1})], \]
which entails that
\[ D(gx_n, gx_{n+1}) \leq \frac{k}{1 - k}(gx_{n-1}, gx_n). \]

Similarly,
\[ D(gx_{n+1}, gx_n) \leq \frac{k}{1 - k}D(gx_n, gx_{n-1}). \]

Hence $(gx_n)$ is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that $(gx_n)$ $D^*$-converges to $x^*$. In other words, there is a $p^* \in X$ such that $(gx_n)$ converges to $g(p^*) = x^*$.

Moreover since
\[ D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq k[D(fx_{n-1}, gp^*) + D(gx_{n-1}, fp^*)], \]
we get that
\[ D(gp^*, fp^*) \leq kD(gp^*, fp^*) \]
which implies that $D(gp^*, fp^*) = 0$.

In the same way, we establish that $D(fp^*, gp^*) = 0$, to then conclude that $fp^* = gp^*$.

We finish the proof by showing that $f$ and $g$ have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now
\[ D(gz^*, gp^*) = D(fz^*, fp^*) \leq k[D(fz^*, gp^*) + D(gz^*, fp^*)] \]
\[ \leq 2kD(gz^*, gp^*), \]
which gives $D(gz^*, gp^*) = 0$. On the other hand, by the same reasoning, it also clear that $D(gp^*, gz^*) = 0$. Therefore $gz^* = gp^*$. From Proposition 1, $f$ and $g$ have a unique common fixed point.

**Theorem 7.** Let $(X, D, \alpha)$ be a $T_0$-quasi-pseudometric type space. Suppose that $f, g : X \rightarrow X$ are mappings such that
\[ D(fx, fy) \leq \lambda D(gx, gy) + \gamma D(fx, gy) \text{ for all } x, y \in X. \]
where $\lambda, \gamma$ are positive constants such that $\lambda + 2\gamma < \min\{1, 1/\alpha\}$. If the range of $g$ contains the range of $f$ and $g(X)$ is bicomplete, then $f$ and $g$ have a unique coincidence point in $X$. Moreover if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.
Proof. Take an arbitrary \( x_0 \in X \). Choose a point \( x_1 \in X \) such that \( f(x_0) = g(x_1) \). This can be done, since \( f(X) \subset g(X) \). Iterating this process, once \( x_n \) is chosen in \( X \), we can obtain \( x_{n+1} \) in \( X \) such that \( f(x_n) = g(x_{n+1}) \). Then
\[
D(gx_n, gx_{n+1}) = D(fx_{n-1}, fx_n) \leq \lambda D(gx_{n-1}, gx_n) + \gamma D(fx_{n-1}, gx_n)
\]
Therefore \((gx_n)\) is a left \( K\)-Cauchy sequence. In a similar manner, we establish that \((gx_n)\) is also a right \( K\)-Cauchy sequence. Hence \((gx_n)\) is a bi-Cauchy sequence. Since \( g(X) \) is bicomplete, there exists \( x^* \in g(X) \) such that \((gx_n)\) converges to \( g(p^*) = x^* \).
Moreover since
\[
D(gx_n, fp^*) = D(fx_{n-1}, fp^*) \leq \lambda D(gx_{n-1}, gp^*) + \gamma D(fx_{n-1}, gp^*)
\]
we get that \( D(gp^*, fp^*) = 0 \). On the other hand, by the same reasoning, it is also clear that \( D(fp^*, gp^*) = 0 \). Hence \( fp^* = gp^* \).
We finish the proof by showing that \( f \) and \( g \) have a unique point of coincidence. For this, assume \( z^* \in X \) is a point such that \( fz^* = gz^* \). Now
\[
D(gz^*, gp^*) = D(fz^*, fp^*) \leq \lambda D(gz^*, gp^*) + \gamma D(fz^*, gp^*)
\]
which gives \( D(gz^*, gp^*) = 0 \). On the other hand, by the same reasoning, it also clear that \( D(gp^*, gz^*) = 0 \). Hence \( gz^* = gp^* \). From Proposition 1, \( f \) and \( g \) have a unique common fixed point.

We now give an example to illustrate Theorems 5, 7.

Example 2. Let \( X = \mathbb{R} \), \( D(x, y) = \max\{x - y, 0\} \) whenever \( x, y \in \mathbb{R} \), \( f(x) = 2x^2 + 4x + 1 \) and \( g(x) = 3x^2 + 6x + 2 \). Then it easy to see that
\[
f(X) = g(X) = [1, \infty) \text{ is bicomplete.}
\]
All the conditions of Theorems 5, 7 are satisfied. Indeed:

- for Theorem 5, take \( k \in \left[\frac{2}{3}, 1\right)\)
- for Theorem 7, take \( \lambda \in \left[\frac{3}{5}, 1\right), \gamma = 0 \).

\( f \) and \( g \) become weakly compatible and we obtain a unique point of coincidence and a unique common fixed point \( -1 = f(-1) = g(-1) \).
Corollary 1. Let \((X, D, \alpha)\) be a \(T_0\)-quasi-pseudometric type space. Suppose that mappings \(f, g : X \to X\) satisfy the contractive condition

\[
D(fx, fy) \leq \alpha[D(gx, gy) + D(fx, fy)] \quad \text{for all } x, y \in X.
\]

where \(0 < \alpha < \min\{1, 1/3\alpha\}\). If the range of \(g\) contains the range of \(f\) and \(g(X)\) is bicomplete, then \(f\) and \(g\) have a unique coincidence point in \(X\). Moreover if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

Theorem 8. Let \((X, D, \alpha)\) be a \(T_0\)-quasi-pseudometric type space. Suppose that \(f, g : X \to X\) are mappings such that

\[
D(fx, fy) \leq \lambda D(gx, gy) + \gamma D(gx, fy) \quad \text{for all } x, y \in X.
\]

where \(\lambda, \gamma\) are positive constants such that \(\lambda+2\gamma < \min\{1, 1/\alpha\}\). If the range of \(g\) contains the range of \(f\) and \(g(X)\) is bicomplete, then \(f\) and \(g\) have a unique coincidence point in \(X\). Moreover if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

Corollary 2. Let \((X, D, \alpha)\) be a \(T_0\)-quasi-pseudometric type space. Suppose that mappings \(f, g : X \to X\) satisfy the contractive condition

\[
D(fx, fy) \leq \lambda[D(gx, gy) + D(gx, fy)] \quad \text{for all } x, y \in X.
\]

where \(0 < \lambda < \min\{1, 1/3\alpha\}\). If the range of \(g\) contains the range of \(f\) and \(g(X)\) is bicomplete, then \(f\) and \(g\) have a unique coincidence point in \(X\). Moreover if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

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