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CONTRA \((m_X, m_Y)\)-SEMICONtinuous FUNCTIONS IN \(m\)-SPACES

Abstract. In this paper, we introduce the notion of contra \((m_X, m_Y)\)-semicontinuous functions between \(m\)-spaces. We obtain many characterizations of these functions and deal with decompositions of the functions and other related functions.

Key words: contra \((m_X, m_Y)\)-semicontinuity, \(m_X\)-semi-closed set, \(m_X\)-semi-open set, minimal structure, minimal space.

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1. Introduction

Generalizations of open sets in a topological space: \(\alpha\)-sets [8], preopen sets [3], semi-open sets [1] and \(\beta\)-open sets etc are very important for generalizing continuity in topological spaces. Various generalizations of continuity are defined and investigated by many authors. As a generalization of the topology, Maki [2] define the notion of minimal structures. A subfamily \(m\) of the power set \(P(X)\) on a nonempty set \(X\) is called a minimal structure [2] if \(\emptyset \in m\) and \(X \in m\). The pair \((X, m)\) is called a minimal space. The elements of \(m\) are said to be \(m\)-open. Recently, several generalizations of \(m\)-open sets have been defined and investigated in [4, 5, 6] and [15]. Quite recently, Sengul and Rosas [14] introduced the notion of contra \((m_X, m_Y)\)-continuity between \(m\)-spaces.

The purpose of the present paper is to introduce and study the notion of contra \((m_X, m_Y)\)-semicontinuous functions between \(m\)-spaces. In Section 3, we obtain many characterizations of contra \((m_X, m_Y)\)-semicontinuity. In Section 4, we deal with decompositions of contra \((m_X, m_Y)\)-semicontinuity and other related functions. The last section gives some properties of strongly \(S - m_X\)-closed spaces.
2. Preliminaries

**Definition 1** ([2, 11]). A subfamily $m_X$ of the power set $P(X)$ of a nonempty set $X$ is called a minimal structure (briefly, $m$-structure) on $X$ if $\emptyset \in m_X$ and $X \in m_X$. The pair $(X, m_X)$ is called a minimal space (briefly, $m$-space). A member of $m_X$ is said to be $m_X$-open and the complement of an $m_X$-open set is said to be $m_X$-closed.

**Definition 2** ([2, 11]). Let $(X, m_X)$ be a minimal space. For a subset $A$ of $X$, the $m_X$-closure of $A$ and the $m_X$-interior of $A$ are defined as follows:

1. $m_X - \text{Cl}(V) = \bigcap \{F : A \subseteq F, X - F \in m_X\}$.
2. $m_X - \text{Int}(V) = \bigcup \{U : U \subseteq A, U \in m_X\}$.

**Lemma 1** ([2, 11]). Let $(X, m_X)$ be a minimal space and $A, B \subseteq X$. Then the followings hold:

1. $m_X - \text{Cl}(\emptyset) = \emptyset$, $m_X - \text{Cl}(X) = X$.
2. $m_X - \text{Int}(\emptyset) = \emptyset$, $m_X - \text{Int}(X) = X$.
3. If $X - A \in m_X$, then $m_X - \text{Cl}(A) = A$.
4. If $A \in m_X$, then $m_X - \text{Int}(A) = A$.
5. $A \subseteq m_X - \text{Cl}(A)$, $m_X - \text{Int}(A) \subseteq A$.
6. $m_X - \text{Cl}(X - A) = X - (m_X - \text{Int}(A))$.
7. $m_X - \text{Int}(X - A) = X - (m_X - \text{Cl}(A))$.
8. $m_X - \text{Cl}(m_X - \text{Cl}(A)) = m_X - \text{Cl}(A)$.
9. $m_X - \text{Int}(m_X - \text{Int}(A)) = m_X - \text{Int}(A)$.
10. If $A \subseteq B$, then $m_X - \text{Cl}(A) \subseteq m_X - \text{Cl}(B)$.
11. If $A \subseteq B$, then $m_X - \text{Int}(A) \subseteq m_X - \text{Int}(B)$.

**Definition 3** ([2]). Let $(X, m_X)$ be a minimal space. The $m$-structure $m_X$ is said to have property $\mathcal{B}$ if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

**Lemma 2** ([11]). Let $(X, m_X)$ be a minimal space and $m_X$ satisfy property of $\mathcal{B}$. For $A \subseteq X$, the followings hold:

1. $A \in m_X$ if and only if $m_X - \text{Int}(A) = A$.
2. $A$ is $m_X$-closed if and only if $m_X - \text{Cl}(A) = A$.
3. $m_X - \text{Int}(A) \in m_X$.
4. $m_X - \text{Cl}(A)$ is $m_X$-closed.

**Lemma 3** ([11]). Let $(X, m_X)$ be a minimal space and $A \subseteq X$. Then $x \in m_X - \text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ such that $x \in U$.

**Definition 4.** Let $(X, m_X)$ be a minimal space. A subset $A$ of $X$ is said to be $m_X$-clopopen if it is $m_X$-open and $m_X$-closed.

**Definition 5.** Let $(X, m_X)$ be a minimal space. A subset $A$ of $X$ is called
(1) an $m_X$-open set [6] if $A \subseteq m_X - \text{Int}(m_X - \text{Cl}(m_X - \text{Int}(A)))$.

(2) an $m_X$-preopen set [4, 13] if $A \subseteq m_X - \text{Int}(m_X - \text{Cl}(A))$.

(3) a $\beta$- $m_X$-open set [7, 15] if $A \subseteq m_X - \text{Cl}(m_X - \text{Int}(m_X - \text{Cl}(A)))$.

**Definition 6 ([5]).** Let $(X, m_X)$ be a minimal space. A subset $A$ of $X$ is called an $m_X$-semiopen set if $A \subseteq m_X - \text{Cl}(m_X - \text{Int}(A))$. The complement of an $m_X$-semiopen set is called an $m_X$-semiclosed set. The family of all $m_X$-semiopen sets in $X$ is denoted by $\text{MSO}(X)$.

**Lemma 4 ([5]).** Let $(X, m_X)$ be a minimal space and $A \subseteq X$. Then

(1) $A$ is an $m_X$-semiclosed set if and only if $m_X - \text{Int}(m_X - \text{Cl}(A)) \subseteq A$.

(2) $\text{MSO}(X)$ is a minimal structure with property $\mathcal{B}$.

**Definition 7 ([5]).** Let $(X, m_X)$ be a minimal space and $A \subseteq X$. The $m_X$-semi-closure of $A$ and the $m_X$-semi-interior of $A$ are defined as follows:

(1) $m_Xs\text{Cl}(A) = \bigcap\{F : A \subseteq F, F$ is $m_X$-semiclosed in $X\}$.

(2) $m_Xs\text{Int}(A) = \bigcup\{U : U \subseteq A, U$ is $m_X$-semiopen in $X\}$.

**Lemma 5.** Let $(X, m_X)$ be a minimal space. For a subset of $A$ of $X$, the following hold:

(1) $A$ is $m_X$-semiopen if and only if $m_Xs\text{Int}(A) = A$.

(2) $A$ is $m_X$-semiclosed if and only if $m_Xs\text{Cl}(A) = A$.

**Proof.** This follows easily from Lemmas 2 and 4. ■

**Definition 8 ([13]).** Let $(X, m_X)$ be a minimal space. Then a subset $A$ of $X$ is said to be $m_X$-gs-closed if $m_Xs\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m_X$.

**Definition 9 ([13]).** Let $(X, m_X)$ be a minimal space. Then $A \subseteq X$ is called an $m_X$-regular open set if $A = m_X - \text{Int}(m_X - \text{Cl}(A))$. Also $A \subseteq X$ is called an $m_X$-regular closed set if $X - A$ is $m_X$-regular open.

If $A$ is $m_X$-closed, then $m_X - \text{cl}(A) = A$ but the converse is not always true. Therefore, $m_X$-regular open (resp. $m_X$-regular closed) is not always $m_X$-open (resp. $m_X$-closed).

**Definition 10 ([12]).** A subset $U$ of a nonempty set $X$ with a minimal structure $m_X$ is said to be $m_X$-compact relative to $(X, m_X)$ if any cover of $U$ by $m_X$-open sets has a finite subcover.

**Definition 11 ([14]).** Let $(X, m_X)$ and $(Y, m_Y)$ be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be contra $(m_X, m_Y)$-continuous if $f^{-1}(V) = m_X - \text{Cl}(f^{-1}(V))$ for every $m_Y$-open set $V$ of $Y$. 
3. Contra \((m_X, m_Y)\)-semi continuous functions

In this section, we introduce the concept of a contra \((m_X, m_Y)\)-semi continuous function between \(m\)-spaces and investigate some characterizations of this continuity.

**Definition 12.** Let \((X, m_X)\) and \((Y, m_Y)\) be two minimal spaces. Then a function \(f : (X, m_X) \to (Y, m_Y)\) is said to be contra \((m_X, m_Y)\)-semi continuous if \(f^{-1}(V)\) is \(m_X\)-semiclosed in \(X\) for every \(m_Y\)-open set \(V\) of \(Y\).

**Lemma 6.** Every contra \((m_X, m_Y)\)-continuous function is contra \((m_X, m_Y)\)-semi continuous.

**Proof.** Let \(f : (X, m_X) \to (Y, m_Y)\) be a contra \((m_X, m_Y)\)-continuous function and \(V\) be any \(m_Y\)-open set of \(Y\). Then \(m_X - \text{Cl}(f^{-1}(V)) = f^{-1}(V)\) and \(m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) = m_X - \text{Int}f^{-1}(V) \subseteq f^{-1}(V)\). Therefore, Lemma 4, \(f^{-1}(V)\) is \(m_X\)-semiclosed and \(f\) is contra \((m_X, m_Y)\)-semi continuous. \(\blacksquare\)

**Remark 1.** The converse of Lemma 6 is not always true as the following example shows.

**Example 1.** Let \(X = \{a, b, c\}\) and \(m_{X_1}, m_{X_2}\) be two minimal structures on \(X\) as follows:

\[
\begin{align*}
m_{X_1} &= \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}, \\
m_{X_2} &= \{\emptyset, X, \{c\}\}.
\end{align*}
\]

Define a function \(f : (X, m_{X_1}) \to (X, m_{X_2})\) as follows:

\[
\begin{align*}
f(a) &= b, \\
f(b) &= c, \\
f(c) &= a.
\end{align*}
\]

Then \(f\) is contra \((m_X, m_Y)\)-semi continuous, but it is not contra \((m_X, m_Y)\)-continuous.

**Theorem 1.** A function \(f : (X, m_X) \to (Y, m_Y)\) is contra \((m_X, m_Y)\)-semi continuous if and only if \(f : (X, \text{MSO}(X)) \to (Y, m_Y)\) is contra \((m_X, m_Y)\)-continuous.

**Proof.** Necessity. Let \(f : (X, m_X) \to (Y, m_Y)\) be contra \((m_X, m_Y)\)-semi continuous and \(V\) be any \(m_Y\)-open set of \(Y\). Then, by hypothesis \(f^{-1}(V)\) is \(m_X\)-semiclosed in \(X\) and, by Lemma 5, \(f^{-1}(V) = m_X s\text{Cl}(f^{-1}(V))\). Therefore, \(f : (X, \text{MSO}(X)) \to (Y, m_Y)\) is contra \((m_X, m_Y)\)-continuous.

Sufficiency. Let \(V\) be any \(m_Y\)-open set of \(Y\). By hypothesis, \(f^{-1}(V) = m_X s\text{Cl}(f^{-1}(V))\) and, by Lemma 5, \(f^{-1}(V) = m_X\)-semi-closed. Therefore, \(f : (X, m_X) \to (Y, m_Y)\) is contra \((m_X, m_Y)\)-semi continuous. \(\blacksquare\)
Definition 13. Let \((X, m_X)\) and \((Y, m_Y)\) be two minimal spaces. Then a function \(f : (X, m_X) \to (Y, m_Y)\) is said to be contra \((m_X, m_Y)\)-semicontinuous at \(x \in X\) if for each \(m_Y\)-closed \(V\) of \(Y\) containing \(f(x)\), there exists an \(m_X\)-semiopen set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq V\).

Theorem 2. Let \((X, m_X)\), \((Y, m_Y)\) be two minimal spaces. A function \(f : (X, m_X) \to (Y, m_Y)\) is contra \((m_X, m_Y)\)-semi continuous if and only if \(f\) is contra \((m_X, m_Y)\)-semicontinuous at each point \(x \in X\).

Proof. Necessity. Let \(x \in X\) and \(V\) be any \(m_Y\)-closed set of \(Y\) containing \(f(x)\). Then \(Y - V\) is \(m_Y\)-open. By hypothesis, \(f^{-1}(Y - V)\) is an \(m_X\)-semiopen subset of \(X\). Thus \(f^{-1}(V)\) is \(m_Y\)-semiopen. Put \(U = f^{-1}(V)\). Then \(x \in U\) and \(f(U) \subseteq V\). This shows that \(f\) is contra \((m_X, m_Y)\)-semicontinuous at each point \(x \in X\).

Sufficiency. Let \(V\) be any \(m_Y\)-open set of \(Y\) and \(x \in f^{-1}(Y - V)\). Then \(f(x) \in Y - V\) and \(Y - V\) is \(m_Y\)-closed. By hypothesis, there exists an \(m_X\)-semiopen set \(U_x\) containing \(x\) such that \(f(U_x) \subseteq Y - V\); hence \(x \in U_x \subseteq f^{-1}(Y - V)\). Therefore, we have \(\cup\{U_x : x \in f^{-1}(Y - V)\} = f^{-1}(Y - V)\). Since \(MSO(X)\) satisfies property \(B\), \(f^{-1}(Y - V)\) is \(m_X\)-semiopen and \(f^{-1}(V)\) is \(m_X\)-semiclosed in \(X\). This shows that \(f\) contra \((m_X, m_Y)\)-semi continuous.

Theorem 3. Let \((X, m_X)\) and \((Y, m_Y)\) be two minimal spaces. For a function \(f : (X, m_X) \to (Y, m_Y)\), the following statements are equivalent:

1. \(f\) is contra \((m_X, m_Y)\)-semi continuous;
2. \(f^{-1}(V)\) is \(m_X\)-semiopen in \(X\) for every \(m_Y\)-closed subset \(V\) of \(Y\);
3. \(m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) = m_X - \text{Int}(f^{-1}(V))\) for every \(m_Y\)-open subset \(V\) of \(Y\);
4. \(m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(V))) = m_X - \text{Cl}(f^{-1}(V))\) for every \(m_Y\)-closed subset \(V\) of \(Y\).

Proof. (1) \(\Rightarrow\) (2). Let \(V\) be any \(m_Y\)-closed set of \(Y\). Then \(Y - V\) is \(m_Y\)-open. Using the hypothesis, \(f^{-1}(Y - V) = X - f^{-1}(V)\) is \(m_X\)-semiclosed in \(X\). As a consequence, \(f^{-1}(V)\) is \(m_X\)-semiopen in \(X\).

(2) \(\Rightarrow\) (3). Let \(V\) be any \(m_Y\)-open set of \(Y\). Then \(Y - V\) is \(m_Y\)-closed. By (2), \(f^{-1}(Y - V)\) is \(m_X\)-semiopen and \(f^{-1}(V)\) is \(m_X\)-semiclosed in \(X\). By Lemma 4, \(m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) \subseteq f^{-1}(V)\) and hence by Lemma 1 \(m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) \subseteq m_X - \text{Int}(f^{-1}(V)) \subseteq m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V)))\). Therefore, we obtain (3).

(3) \(\Rightarrow\) (4). It is clear from the complement of (3).

(4) \(\Rightarrow\) (1). Let \(V\) be any \(m_Y\)-open subset of \(Y\). Then \(Y - V\) is \(m_Y\)-closed. By hypothesis,

\[
m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(Y - V))) = m_X - \text{Cl}(f^{-1}(Y - V)).
\]
Then we obtain that
\[ m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(V))) = m_X - \text{Int}(f^{-1}(V)) \subseteq f^{-1}(V). \]

By Lemma 4, \( f^{-1}(V) \) is \( m_X \)-semiclosed in \( X \).

**Theorem 4.** Let \((X, m_X), (Y, m_Y)\) be two minimal spaces and \( m_Y \) satisfy property \( \mathcal{B} \). For a function \( f : (X, m_X) \rightarrow (Y, m_Y) \), the following statements are equivalent:

1. \( f \) is contra \((m_X, m_Y)\)-semi continuous;
2. \( f^{-1}(B) \) is \( m_X \)-semiopen in \( X \) for every \( m_Y \)-closed set \( B \) in \( Y \);
3. \( f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B)))) \) for every subset \( B \) in \( Y \);
4. \( m_X - \text{Int}(m_X - \text{Cl}(f^{-1}(m_Y - \text{Int}(B)))) \subseteq f^{-1}(B) \) for every subset \( B \) in \( Y \);
5. \( A \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f(A)))))) \) for every subset \( A \) in \( X \).

**Proof.** (1) \( \Leftrightarrow \) (2). It is obvious from Theorem 3.

(2) \( \Rightarrow \) (3). Let \( B \subseteq Y \). Then \( m_Y - \text{Cl}(B) \) is an \( m_Y \)-closed set in \( Y \) since \( m_Y \) satisfies property \( \mathcal{B} \). By (2), \( f^{-1}(m_Y - \text{Cl}(B)) \) is \( m_X \)-semiopen in \( X \). Therefore, \( f^{-1}(m_Y - \text{Cl}(B)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B)))) \). As a consequence, \( f^{-1}(B) \subseteq f^{-1}(m_Y - \text{Cl}(B)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B)))) \).

(3) \( \Leftrightarrow \) (4). It is clear from the complement.

(4) \( \Rightarrow \) (5). Let \( A \subseteq X \). Then \( f(A) \subseteq Y \). By (3), \( A \subseteq f^{-1}(f(A)) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f(A))))) \).

(5) \( \Rightarrow \) (2). Let \( B \) be any \( m_Y \)-closed set in \( Y \). Then \( f^{-1}(B) \subseteq X \). By (5), \( f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(f^{-1}(B)))))) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(m_Y - \text{Cl}(B)))) \). Then we obtain

\[ f^{-1}(B) \subseteq m_X - \text{Cl}(m_X - \text{Int}(f^{-1}(B))) \]

since \( B \) is \( m_Y \)-closed in \( Y \). As a consequence, \( f^{-1}(B) \) is \( m_X \)-semiopen in \( X \).

**Theorem 5.** Let \((X, m_X), (Y, m_Y)\) be two minimal spaces and \( m_X, m_Y \) satisfy property \( \mathcal{B} \). For a function \( f : (X, m_X) \rightarrow (Y, m_Y) \), the following statements are equivalent:

1. \( f \) is contra \((m_X, m_Y)\)-semi continuous;
2. \( f^{-1}(V) \) is \( m_X \)-semiopen in \( X \) for every \( m_Y \)-closed subset \( V \) of \( Y \);
3. There exists an \( m_X \)-semiclosed set \( U \) such that \( x \notin U \) and \( f^{-1}(V) \subseteq U \) for each \( x \in X \) and each \( m_Y \)-open \( V \) with \( f(x) \notin V \).
(4) \( f^{-1}(F) \subseteq m_XsInt(f^{-1}(F)) \) for any \( m_Y \)-closed set \( F \) in \( Y \);
(5) \( m_XsCl(f^{-1}(F)) \subseteq f^{-1}(F) \) for any \( m_Y \)-open set \( F \) in \( Y \);
(6) \( m_XsCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F)) \) for any subset \( F \subseteq Y \);
(7) \( f^{-1}(m_Y - Cl(F)) \subseteq m_XsInt(f^{-1}(m_Y - Cl(F))) \) for any subset \( F \subseteq Y \).

**Proof.** (1) \( \Leftrightarrow \) (2) is already shown in Theorem 3.

(1) \( \Rightarrow \) (3). Let \( x \in X \) and \( V \) be any \( m_Y \)-open subset of \( Y \) with \( f(x) \notin V \). Then \( f^{-1}(V) \) is \( m_X \)-semiclosed. Put \( U = f^{-1}(V) \). Then \( f^{-1}(V) \subseteq U \) and \( x \notin U \).

(3) \( \Rightarrow \) (1). Let \( V \) be any \( m_Y \)-open subset of \( Y \). For each \( x \in f^{-1}(Y - V) \), \( f(x) \notin V \). By hypothesis, there exists an \( m_X \)-semiclosed set \( U_x \) such that \( x \notin U_x \) and \( f^{-1}(V) \subseteq U_x \). Then \( x \in X - U_x \subseteq X - f^{-1}(V) = f^{-1}(Y - V) \).

We obtain

\[
\bigcup_{x \in f^{-1}(Y - V)} \{ x \} \subseteq \bigcup_{x \in f^{-1}(Y - V)} (X - U_x) \subseteq f^{-1}(Y - V).
\]

Hence \( f^{-1}(Y - V) = \bigcup_{x \in f^{-1}(Y - V)} (X - U_x) \) is \( m_X \)-semiopen. Thus \( f^{-1}(V) \) is \( m_X \)-semiclosed. As a consequence, \( f \) is contra \((m_X, m_Y)\)-semi continuous.

(1) \( \Rightarrow \) (4). Let \( F \) be any \( m_Y \)-closed subset of \( Y \). For each \( x \in f^{-1}(F) \), \( f(x) \in F \). By Theorem 2, there exists an \( m_X \)-semiopen set \( U \) such that \( x \in U \) and \( f(U) \subseteq F \). Since \( x \in U \subseteq f^{-1}(F) \), we obtain \( x \in m_XsInt(f^{-1}(F)) \).

As a consequence, \( f^{-1}(F) \subseteq m_XsInt(f^{-1}(F)) \).

(4) \( \Rightarrow \) (5). It is obvious from taking the complement of (4).

(5) \( \Rightarrow \) (6). Let \( F \) be any subset of \( Y \). Since \( m_Y \) satisfies property \( \mathcal{B} \), \( m_Y - Int(F) \) is an \( m_Y \)-open subset of \( Y \) and by (5), we obtain

\[
m_XsCl(f^{-1}(m_Y - Int(F))) \subseteq f^{-1}(m_Y - Int(F)).
\]

(6) \( \Rightarrow \) (7). It is clear from the complement of (6).

(7) \( \Rightarrow \) (1). Let \( V \) be any \( m_Y \)-open subset of \( Y \). Then \( Y - V \) is \( m_Y \)-closed. By (7), \( X - f^{-1}(V) = f^{-1}(Y - V) = f^{-1}(m_Y - Cl(Y - V)) \subseteq m_XsInt(f^{-1}(m_Y - Cl(Y - V))) = m_XsInt(f^{-1}(Y - V)) = X - m_XsCl(f^{-1}(V)) \). Therefore, \( m_X - sCl(f^{-1}(V)) \subseteq f^{-1}(V) \) and hence \( m_X - sCl(f^{-1}(V)) = f^{-1}(V) \). Since \( m_X \) satisfies property \( \mathcal{B} \), \( f^{-1}(V) \) is \( m_X \)-semiclosed in \( X \). As a consequence, \( f \) is contra \((m_X, m_Y)\)-semi continuous. \( \blacksquare \)

4. **Decompositions of contra \((m_X, m_Y)\)-semincontinuity**

In this section, we obtain decompositions of contra \((m_X, m_Y)\)-semincontinuous functions and other related functions.
**Definition 14.** Let $\langle X, m_X \rangle$ be a minimal space. A subset $A$ of $X$ is called

1. an $m_X$-semi-regular set if $A$ is both $m_X$-semiopen and $m_X$-semiclosed.
2. an $m_X$-$B$-set if $A = U \cap V$, where $U \in m_X$ and $V$ is $m_X$-semiclosed.

**Lemma 7.** Let $\langle X, m_X \rangle$ be a minimal space and $A \subseteq X$. Then the following conditions are equivalent:

1. $A$ is $m_X$-semi-regular;
2. $A$ is both $\beta m_X$-open and $m_X$-semiclosed.

**Proof.** It is obvious by Lemma 4.

**Remark 2.** A $\beta m_X$-open set and an $m_X$-semiclosed set are independent of each other as the following examples show.

**Example 2.** Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, b\}$ is an $m_X$-open set and hence $\beta m_X$-open, but it is not an $m_X$-semiclosed set.

**Example 3.** Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$. Then $A = \{a, b\}$ is an $m_X$-closed set and hence $m_X$-semiclosed set, but it is not a $\beta m_X$-open set.

**Lemma 8.** Let $\langle X, m_X \rangle$ be a minimal space and $m_X$ satisfy property $\mathcal{B}$. Then for a subset $A$ of $X$, the following conditions are equivalent:

1. $A$ is both $m_X$-open and $m_X$-semiclosed;
2. $A$ is both $\alpha m_X$-open and $m_X$-semiclosed;
3. $A$ is both $m_X$-preopen and $m_X$-semiclosed.

**Proof.** It is clear.

**Remark 3.** An $m_X$-preopen set and an $m_X$-semiclosed set are independent of each other as the following example shows.

**Example 4.** Consider Example 2, then the set $A = \{a, b\}$ is an $m_X$-preopen set, but it is not $m_X$-semiclosed. Also in Example 3, the set $A$ is an $m_X$-semiclosed set, but it is not an $m_X$-preopen set.

**Lemma 9.** Let $\langle X, m_X \rangle$ be a minimal space and $A \subseteq X$. If $A$ is both $\beta m_X$-open and $m_X$-closed, then it is $m_X$-regular closed.

**Proof.** It is an immediate result.

**Remark 4.** A $\beta m_X$-open set and an $m_X$-closed set are independent of each other as the following example shows.
Example 5. Consider Example 2, then the set $A = \{a, b\}$ is a $\beta m_X$-open set, but it is not an $m_X$-closed set. Also in Example 3, the set $A$ is an $m_X$-closed set, but it is not a $\beta m_X$-open set.

Definition 15. Let $(X, m_X)$ and $(Y, m_Y)$ be two minimal spaces. Then a function $f : (X, m_X) \to (Y, m_Y)$ is said to be

1. $(m_X, m_Y)$-perfectly continuous if $f^{-1}(V)$ is $m_X$-clopen in $X$ for every $m_Y$-open set $V$ of $Y$,
2. $(m_X, m_Y)$-completely continuous if $f^{-1}(V)$ is $m_X$-regular open in $X$ for every $m_Y$-open set $V$ of $Y$,
3. $(m_X, m_Y)$-semi-regular continuous (briefly, $(m_X, m_Y)$-SR-continuous) if $f^{-1}(V)$ is $m_X$-semi-regular open in $X$ for every $m_Y$-open set $V$ of $Y$,
4. $(m_X, m_Y)$-regular closed continuous (briefly, $(m_X, m_Y)$-RC-continuous) if $f^{-1}(V)$ is $m_X$-regular closed in $X$ for every $m_Y$-open set $V$ of $Y$,
5. $(m_X, m_Y)$-B-continuous if $f^{-1}(V)$ is an $m_X$-B-set in $X$ for every $m_Y$-open set $V$ of $Y$.

Definition 16 ([7]). Let $m_X, m_Y$ be two minimal structures. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M - \beta$-continuous if $f^{-1}(V)$ is $\beta m_X$-open in $X$ for every $m_Y$-open set $V$ of $Y$.

Theorem 6. For a function $f : (X, m_X) \to (Y, m_Y)$, the following statements are equivalent:

1. $f$ is $(m_X, m_Y)$-SR-continuous;
2. $f$ is $M - \beta$-continuous and contra $(m_X, m_Y)$-semi continuous.

Proof. It is an immediate result of Lemma 7.

Definition 17 ([4]). Let $m_X, m_Y$ be two minimal structures. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M$-pre continuous if $f^{-1}(V)$ is $m_X$-preopen in $X$ for every $m_Y$-open set $V$ of $Y$.

Theorem 7. If a function $f : (X, m_X) \to (Y, m_Y)$ is $M$-pre continuous and contra $(m_X, m_Y)$-semi continuous, it is $(m_X, m_Y)$-completely continuous.

Proof. It is clear from the fact that every $m_X$-preopen and $m_X$-semiclosed set is $m_X$-regular open.

Theorem 8. If a function $f : (X, m_X) \to (Y, m_Y)$ is $M - \beta$-continuous and contra $(m_X, m_Y)$-continuous, it is $(m_X, m_Y)$-RC-continuous.

Proof. It is obvious from Lemma 9.
**Definition 18.** A function \( f : (X, m_X) \to (Y, m_Y) \) is said to be contra \((m_X, m_Y)\)-gs-continuous if \( f^{-1}(V) \) is \( m_X \)-gs-closed in \( X \) for every \( m_Y \)-open set \( V \) of \( Y \).

**Theorem 9.** For a function \( f : (X, m_X) \to (Y, m_Y) \), the following statements are equivalent:

1. \( f \) is contra \((m_X, m_Y)\)-semi continuous;
2. \( f \) is \((m_X, m_Y)\)-B-continuous and contra \((m_X, m_Y)\)-gs-continuous.

**Proof.** (1) \( \Rightarrow \) (2). It is clear.

(2) \( \Rightarrow \) (1). Let \( V \) be any \( m_Y \)-open set of \( Y \). Since \( f \) is \((m_X, m_Y)\)-B-continuous, \( f^{-1}(V) = U \cap F \), where \( U \in m_X \) and \( F \) is \( m_X \)-semiclosed in \( X \). Then \( f^{-1}(V) \subseteq U \) and \( U \in m_X \). \( f^{-1}(V) \) is \( m_X \)-gs-closed and since \( f \) is contra \((m_X, m_Y)\)-gs-continuous, \( m_X \) \( sCl(f^{-1}(V)) = m_X - Int(m_X - Cl(m_X \ sCl(f^{-1}(V)))) \subseteq m_X - Int(m_X - Cl(m_X \ sCl(f^{-1}(V)))) \subseteq m_X \ sCl(f^{-1}(V)) \subseteq U \). On the other hand, \( F \) is \( m_X \)-semiclosed and by Lemma 4 \( m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq m_X - Int(m_X - Cl(f^{-1}(V))) \subseteq U \). As a consequence, \( f^{-1}(V) \) is \( m_X \)-semiclosed.

**Remark 5.** The notions of \((m_X, m_Y)\)-B-continuity and contra \((m_X, m_Y)\)-gs-continuity are independent of each other as shown by the following example.

**Example 6.** Let \( X = \{1, 2\} \), \( Y = \{a, b\} \), \( m_X = \{\emptyset, X, \{2\}\} \) and \( m_Y = \{\emptyset, Y\} \). Let \( f : (X, m_X) \to (X, m_X) \) be the identity function. Then \( f \) is \((m_X, m_Y)\)-B-continuous but it is not contra \((m_X, m_Y)\)-gs-continuous. Also, let \( g : (Y, m_Y) \to (X, m_X) \) be a function defined as follows:

\[
g(a) = 1, \quad g(b) = 2.
\]

Then \( g \) is contra \((m_X, m_Y)\)-gs-continuous, but it is not \((m_X, m_Y)\)-B-continuous.

**Corollary 1.** For a function \( f : (X, m_X) \to (Y, m_Y) \), the following statements are equivalent:

1. \( f \) is \((m_X, m_Y)\)-SR-continuous;
2. \( f \) is \( M-\beta \)-continuous, \((m_X, m_Y)\)-B-continuous and contra \((m_X, m_Y)\)-gs-continuous.

**Proof.** It is obvious from Theorems 6 and 9.

**Remark 6.** The function \( f : (X, m_X) \to (X, m_X) \) in Example 6 is \((m_X, m_Y)\)-pre continuous, but it is not contra \((m_X, m_Y)\)-gs-continuous. Also, the function \( g : (Y, m_Y) \to (X, m_X) \) in Example 6 is \((m_X, m_Y)\)-pre continuous, but it is not \((m_X, m_Y)\)-B-continuous.
**Remark 7.** We obtain the following diagram which shows the relationships between contra \((m_X, m_Y)\)-semicontinuous functions and other related functions.

\[
\text{DIAGRAM}
\begin{align*}
&\text{m-regular closed } C \\
\downarrow & \\
&\text{m-complete } C \rightarrow M\text{-pre } C \rightarrow M - \beta - C \rightarrow \text{contra-}m - C \\
\uparrow & \\
&\text{m-perfect } C \rightarrow \text{m-semi-regular } C \rightarrow \text{contra } m\text{-semi } C \\
\downarrow & \\
&\text{m-B-C } \rightarrow \text{contra-mgs-C}
\end{align*}
\]

In the diagram, \(C\) denotes continuity and \(m\) means \((m_X, m_Y)\).

5. **Strongly \(S - m_X\)-closed spaces**

**Definition 19.** A minimal space \((X, m_X)\) is said to be

1. \(m_X\)-semi-compact if there exists a finite subset \(J\) of \(I\) such that \(X = \bigcup\{U_i : i \in J\}\) for every \(m_X\)-semiopen cover \(\{U_i : i \in I\}\) of \(X\),
2. \(m_X\)-s-closed if there exists a finite subset \(J\) of \(I\) such that \(X = \bigcup\{m_Xs\text{Cl}(U_i) : i \in J\}\) for every \(m_X\)-semiopen cover \(\{U_i : i \in I\}\) of \(X\),
3. \(m_X\)-S-closed if there exists a finite subset \(J\) of \(I\) such that \(X = \{m_X - \text{Cl}(U_i) : i \in J\}\) for every \(m_X\)-semiopen cover \(\{U_i : i \in I\}\) of \(X\),
4. \([14]\) \(m_X\)-nearly compact if there exists a finite subset \(J\) of \(I\) such that \(X = \bigcup\{m_X - \text{Int}(m_X - \text{Cl}(U_i)) : i \in J\}\) for every \(m_X\)-open cover \(\{U_i : i \in I\}\) of \(X\),
5. \([9]\) \(m_X\)-closed if there exists a finite subset \(J\) of \(I\) such that \(X = \bigcup\{m_X - \text{Cl}(U_i) : i \in J\}\) for every \(m_X\)-open cover \(\{U_i : i \in I\}\) of \(X\),
6. \([10]\) strongly \(S\)-\(m_X\)-closed if every \(m_X\)-closed cover of \(X\) has a finite subcover,
7. \(m_X\)-mildly compact if every \(m_X\)-clopen cover of \(X\) has a finite subcover.

We obtain the following diagram:

\[
\text{DIAGRAM}
\begin{align*}
&m_X\text{-semi-compact } \rightarrow m_X\text{-s-closed } \rightarrow m_X\text{-S-closed strongly } S\text{-}m_X\text{-closed} \\
\downarrow & \quad \downarrow \quad \downarrow \\
&m_X\text{-compact } \rightarrow m_X\text{-nearly compact } \rightarrow m_X\text{-closed } \rightarrow m_X\text{-mildly compact}
\end{align*}
\]
Theorem 10. Let $(X, m_X), (Y, m_Y)$ be two minimal spaces and a function $f : (X, m_X) \to (Y, m_Y)$ be a surjection. If one of the following statements holds, then $(Y, m_Y)$ is strongly $S_{m_Y}$-closed.

(1) $f$ is contra $(m_X, m_Y)$-semi continuous and $(X, m_X)$ is $m_X$-semi-compact,

(2) $f$ is $(m_X, m_Y)$-perfectly continuous and $(X, m_X)$ is $m_X$-mildly compact.

Proof. Suppose (2) holds: Let $\{U_i : i \in I\}$ be an $m_Y$-closed cover of $Y$. $\{f^{-1}(U_i) : i \in I\}$ is an $m_X$-clopen cover of $X$ since $f$ is $(m_X, m_Y)$-perfectly continuous. Then there exists a finite $J \subseteq I$ such that $X = \bigcup_{i \in J} f^{-1}(U_i)$ as $(X, m_X)$ is $m_X$-mildly compact. Hence $Y = \bigcup_{i \in J} U_i$. As a consequence, $(Y, m_Y)$ is strongly $S_{m_Y}$-closed. ■

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