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HERMITE-HADAMARD TYPE INEQUALITIES FOR MT$_m$-PREINVEX FUNCTIONS

ABSTRACT. In the present paper, the notion of MT$_m$-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving MT$_m$-preinvex functions along with beta function are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for MT$_m$-preinvex functions via classical integrals and Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These results not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

KEY WORDS: Hermite-Hadamard type inequality, MT-convex function, Hölder’s inequality, power mean inequality, fractional integral, $m$-invex, $P$-function.

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1. Introduction and preliminaries

The following notations are used throughout this paper. We use $I$ to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and $I^\circ$ to denote the interior of $I$. For any subset $K \subseteq \mathbb{R}^n$, $K^\circ$ is used to denote the interior of $K$. $\mathbb{R}^n$ is used to denote a generic $n$-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.** Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]
In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [9], [13], [6], [21], [5], [11], [10], [4], [17], [3]) and the references cited therein. In (see [19], [14]) and the references cited therein, Tunç and Yildirim defined the following so-called MT-convex function:

**Definition 1.** A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:

\[
    f(tx + (1 - t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).
\]

Fractional calculus (see [13]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

**Definition 2.** Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

\[
    J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
    J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (t-x)^{\alpha-1} f(t)dt, \quad b > x,
\]

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

**Definition 3** (see [7]). A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be $P$-function or $P$-convex, if

\[
    f(tx + (1 - t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0, 1].
\]

**Definition 4** (see [1]). A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$. 
Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details please see (see [1],[20]) and the references therein.

**Definition 5** (see [16]). The function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect $\eta$, if for every $x, y \in K$ and $t \in [0, 1]$, we have

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x - a)^p(b - x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^*|f|,$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^*|f|$ (see [18]).

Recently, Liu (see [12]) obtained several integral inequalities for the left-hand side of (3) under the Definition 3 of $P$-function. Also in (see [15]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $MT_m$-preinvex function is introduced and some new integral inequalities for the left-hand side of (3) involving $MT_m$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for $MT_m$-preinvex functions via classical integrals are given. In Section 4, some generalizations of Hermite-Hadamard type inequalities for $MT_m$-preinvex functions via fractional integrals are given. In Section 5, some applications to special means are given. These results given in Sections 3-4 not only extend the results appeared in the literature (see [13]), but also provide new estimates on these types.

2. New integral inequalities for $MT_m$-preinvex functions

**Definition 6** (see [8]). A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0, 1) \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

**Remark 1.** In Definition 6, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.

We next give new definition, to be referred as $MT_m$-preinvex function.
Definition 7. Let \( K \subseteq \mathbb{R}^n \) be an open \( m \)-invex set with respect to \( \eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n \). For \( f : K \rightarrow \mathbb{R} \) and any fixed \( m \in (0, 1] \), if
\[
 f(my + t\eta(x, y, m)) \leq \frac{m\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y),
\]
is valid for all \( x, y \in K \) and \( t \in (0, 1) \), then we say that \( f(x) \) belong to the class of \( MT_m(K) \) with respect to \( \eta \).

Remark 2. In Definition 7, it is worthwhile to note that the class \( MT_m(K) \) is a generalization of the class \( MT(I) \) given in Definition 1 on \( K = I \) with respect to \( \eta(x, y, 1) = x - y \) and \( m = 1 \).

Example 1. \( f, g : (1, \infty) \rightarrow \mathbb{R}, f(x) = x^p, g(x) = (1 + x)^p, p \in (0, \frac{1}{100}) ; h : [1, 3/2] \rightarrow \mathbb{R}, h(x) = (1 + x^2)^k, k \in (0, \frac{1}{100}) \), are simple examples of the new class of \( MT_m \)-preinvex functions with respect to \( \eta(x, y, m) = x - my \) for any fixed \( m \in (0, 1] \), but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving \( MT_m \)-preinvex functions along with beta function, we need the following new lemma:

Lemma 1. Let \( f : K = [ma, ma + \eta(b, a, m)] \rightarrow \mathbb{R} \) be a continuous function on the interval of real numbers \( K^o \) with \( a < b \) and \( ma < ma + \eta(b, a, m) \). Then for any fixed \( m \in (0, 1] \) and any fixed \( p, q > 0 \), we have
\[
 \int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b, a, m) - x)^q f(x)dx = \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(ma + t\eta(b, a, m))dt.
\]

Proof. It is easy to observe that
\[
 \int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b, a, m) - x)^q f(x)dx = \eta(b, a, m) \int_0^1 (ma + t\eta(b, a, m) - ma)^p (ma + \eta(b, a, m) - ma)^q f(ma + t\eta(b, a, m))dt
\]
\[
 = \eta(b, a, m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(ma + t\eta(b, a, m))dt.
\]

The following definition will be used in the sequel.
Definition 8. The Euler Beta function is defined for \(x, y > 0\) as
\[
\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
\]

Theorem 2. Let \(f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbb{R}\) be a continuous function on the interval of real numbers \(K^\circ, a < b\) with \(ma < ma + \eta(b, a, m)\). If \(|f|\) is a \(MT_m\)-preinvex function on \(K\) for any fixed \(m \in (0, 1]\), then for any fixed \(p, q > 0\), we have
\[
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) dx
\leq \frac{m}{2} \eta(b,a,m)^{p+q+1} \left[ |f(a)|\beta \left( p + \frac{1}{2}, q + \frac{3}{2} \right) + |f(b)|\beta \left( p + \frac{3}{2}, q + \frac{1}{2} \right) \right].
\]

Proof. Since \(|f|\) is a \(MT_m\)-preinvex function on \(K\), we have
\[
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) dx
\leq \eta(b,a,m)^{p+q+1} \int_0^1 t^p (1-t)^q \left| f(ma + t\eta(b,a,m)) \right| dt
\leq \eta(b,a,m)^{p+q+1} \int_0^1 t^p (1-t)^q \left[ \frac{m\sqrt{t}}{2\sqrt{1-t}} |f(b)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f(a)| \right] dt
\leq \frac{m}{2} \eta(b,a,m)^{p+q+1} \left[ |f(a)|\beta \left( p + \frac{1}{2}, q + \frac{3}{2} \right) + |f(b)|\beta \left( p + \frac{3}{2}, q + \frac{1}{2} \right) \right].
\]

Theorem 3. Let \(f : K = [ma, ma + \eta(b,a,m)] \longrightarrow \mathbb{R}\) be a continuous function on the interval of real numbers \(K^\circ, a < b\) with \(ma < ma + \eta(b,a,m)\). Let \(k > 1\) and \(|f|^{\frac{k}{k-1}}\) be a \(MT_m\)-preinvex function on \(K\) for any fixed \(m \in (0, 1]\). Then for any fixed \(p, q > 0\), we have
\[
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) dx
\leq \left( \frac{m\pi}{4} \right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{k}}
\times \left( |f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}.
\]
Proof. Since $|f|^{\frac{k}{k-1}}$ is a $MT_m$-preinvex function on $K$, combining with Lemma 1 and Hölder inequality for all $t \in (0, 1)$ and for any fixed $m \in (0, 1]$, we get

$$
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) \, dx \\
\leq \eta(b,a,m)^{p+q+1} \left[ \int_0^1 t^{kp}(1-t)^{kq} \, dt \right]^{\frac{1}{k}} \\
\times \left[ \int_0^1 |f(ma + t\eta(b,a,m))|^{\frac{k}{k-1}} \, dt \right]^{\frac{k-1}{k}} \\
\leq \eta(b,a,m)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{k}} \\
\times \left[ \int_0^1 \left( \frac{\sqrt{2}m}{2 \sqrt{1-t}} |f(b)|^{\frac{k}{k-1}} + \frac{\sqrt{2}m}{2 \sqrt{1-t}} |f(a)|^{\frac{k}{k-1}} \right) \, dt \right]^{\frac{k-1}{k}} \\
= \left( \frac{m\pi}{4} \right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{k}} \\
\times \left( |f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}.
$$

\[ \blacksquare \]

Theorem 4. Let $f : K = [ma, ma + \eta(b,a,m)] \to \mathbb{R}$ be a continuous function on the interval of real numbers $K \circ \circ, a < b$ with $ma < ma + \eta(b,a,m)$. Let $l \geq 1$ and $|f|^l$ be a $MT_m$-preinvex function on $K$ for any fixed $m \in (0, 1]$. Then for any fixed $p,q > 0$, we have

$$
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) \, dx \\
\leq \left( \frac{m}{2} \right) \eta(b,a,m)^{p+q+1} \left[ \beta(p + 1, q + 1) \right]^{\frac{1}{l+1}} \\
\times \left[ |f(a)|^l \beta \left( p + \frac{1}{2}, q + \frac{3}{2} \right) + |f(b)|^l \beta \left( p + \frac{3}{2}, q + \frac{1}{2} \right) \right]^{\frac{1}{l}}.
$$

Proof. Since $|f|^l$ is a $MT_m$-preinvex function on $K$, combining with Lemma 1 and Hölder inequality for all $t \in (0, 1)$ and for any fixed $m \in (0, 1]$, we get

$$
\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x) \, dx
$$
\begin{equation}
\eta(b, a, m)p + q + 1 \int_0^1 \left[t^p (1 - t)^q \right]^{\frac{1}{q+1}}
\times \left[t^p (1 - t)^q \right]^{\frac{1}{q+1}} f(ma + t\eta(b, a, m)) dt
\leq \eta(b, a, m)p + q + 1 \left[ \int_0^1 t^p (1 - t)^q dt \right]^{\frac{1}{q+1}}
\times \left[ \int_0^1 t^p (1 - t)^q |f(ma + t\eta(b, a, m))| dt \right]^{\frac{1}{q+1}}
\leq \eta(b, a, m)p + q + 1 \beta(p + 1, q + 1) \left[ \beta(p + 1, q + 1) \right]^{\frac{1}{q+1}}
\times \left[ \int_0^1 t^p (1 - t)^q \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} |f(b)| + \frac{m\sqrt{1 - t}}{2\sqrt{t}} |f(a)| \right) dt \right]^{\frac{1}{q+1}}
= \left( \frac{m}{2} \right)^{\frac{1}{q+1}} \eta(b, a, m)p + q + 1 \beta(p + 1, q + 1) \left[ \beta(p + 1, q + 1) \right]^{\frac{1}{q+1}}
\times \left[ |f(a)|^{\frac{1}{q+1}} \beta \left( p + \frac{1}{2}, q + \frac{3}{2} \right) + |f(b)|^{\frac{1}{q+1}} \beta \left( p + \frac{3}{2}, q + \frac{1}{2} \right) \right]^{\frac{1}{q+1}}.
\end{equation}

**Remark 3.** In Theorem 4, if we choose \( l = 1 \), we get Theorem 2.

### 3. Hermite-Hadamard type classical integral inequalities for \(MT_m\)-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for \(MT_m\)-preinvex functions via classical integrals, we need the following new lemma:

**Lemma 2.** Let \( K \subseteq \mathbb{R} \) be an open \( m\)-invex subset with respect to \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( m \in (0, 1] \) and let \( a, b \in K \), \( a < b \) with \( ma < ma + \eta(b, a, m) \). Assume that \( f : K \rightarrow \mathbb{R} \) is a differentiable function on \( K^\circ \) and \( f' \) is integrable on \([ma, ma + \eta(b, a, m)]\). Then, for each \( x \in [ma, ma + \eta(b, a, m)] \), we have

\begin{equation}
\frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} - \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma + \eta(x, a, m)} f(u) du - \int_{mb}^{mb + \eta(x, b, m)} f(u) du \right]
\end{equation}
\[
\begin{align*}
&= \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \int_0^1 (t - 1) f'(ma + t\eta(x, a, m)) dt \\
&\quad + \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \int_0^1 (1 - t) f'(mb + t\eta(x, b, m)) dt.
\end{align*}
\]

**Proof.** Denote
\[
I = \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \int_0^1 (t - 1) f'(ma + t\eta(x, a, m)) dt \\
+ \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \int_0^1 (1 - t) f'(mb + t\eta(x, b, m)) dt.
\]

Integrating by parts, we get
\[
I = \frac{\eta(x, a, m)^2}{\eta(b, a, m)} \left[ (t - 1) \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} \bigg|_0^1 - \int_0^1 \frac{f(ma + t\eta(x, a, m))}{\eta(x, a, m)} dt \right] \\
+ \frac{\eta(x, b, m)^2}{\eta(b, a, m)} \left[ (1 - t) \frac{f(mb + t\eta(x, b, m))}{\eta(x, b, m)} \bigg|_0^1 + \int_0^1 \frac{f(mb + t\eta(x, b, m))}{\eta(x, b, m)} dt \right] \\
= \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \\
\quad - \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma+\eta(x,a,m)} f(u) du - \int_{mb}^{mb+\eta(x,b,m)} f(u) du \right].
\]

**Remark 4.** Clearly, if we choose \( m = 1 \) and \( \eta(x, y, 1) = x - y \) in Lemma 2, we get (see [9], Lemma 1).

Using the Lemma 2 the following results can be obtained.

**Theorem 5.** Let \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invex subset with respect to \( \eta : A \times A \times (0, 1) \rightarrow \mathbb{R}_0 \) for any fixed \( m \in (0, 1] \) and let \( a, b \in A, a < b \) with \( ma < ma + \eta(b, a, m) \). Assume that \( f : A \rightarrow \mathbb{R} \) is a differentiable function on \( A^o \). If \( |f'| \) is a \( MT_m \)-preinvex function on \( [ma, ma + \eta(b, a, m)] \) and \( |f'(x)| \leq M \), then for each \( x \in [ma, ma + \eta(b, a, m)] \), we have

\[
\begin{align*}
&\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \\
&\quad - \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma+\eta(x,a,m)} f(u) du - \int_{mb}^{mb+\eta(x,b,m)} f(u) du \right] \right| \\
&\leq \frac{Mm\pi}{4|\eta(b, a, m)|} \left[ \eta(x, a, m)^2 + \eta(x, b, m)^2 \right].
\end{align*}
\]
Proof. Using Lemma 2, $MT_m$-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

\[
\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right| \\
- \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma + \eta(x, a, m)} f(u)du - \int_{mb}^{mb + \eta(x, b, m)} f(u)du \right] \\
\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 |t - 1||f'(ma + t\eta(x, a, m))|dt \\
+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 |1 - t||f'(mb + t\eta(x, b, m))|dt \\
\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 (1 - t) \left[ \frac{m\sqrt{t}}{2\sqrt{1 - t}}|f'(x)| \\
+ \frac{m\sqrt{1 - t}}{2\sqrt{t}}|f'(a)| \right] dt \\
+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 (1 - t) \left[ \frac{m\sqrt{t}}{2\sqrt{1 - t}}|f'(x)| + \frac{m\sqrt{1 - t}}{2\sqrt{t}}|f'(b)| \right] dt \\
\leq \frac{Mm\pi}{4|\eta(b, a, m)|} \left[ \eta(x, a, m)^2 + \eta(x, b, m)^2 \right].
\]

\[\blacksquare\]

Remark 5. In Theorem 5, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 2.2).

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 6. Let $A \subseteq \mathbb{R}_0$ be an open $m$-invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0$ for any fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on $A^\circ$. If $|f'|^q$ is a $MT_m$-preinvex function on $[ma, ma + \eta(b, a, m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have

\[
\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right| \\
- \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma + \eta(x, a, m)} f(u)du - \int_{mb}^{mb + \eta(x, b, m)} f(u)du \right] \\
\leq \frac{M}{(p + 1)^{1/p}} \left( \frac{m\pi}{2} \right)^{\frac{1}{q}} \left[ \frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\]
Proof. Suppose that \( q > 1 \). Using Lemma 2, \( MT_m \)-preinvexity of \( |f'|^q \), Hölder inequality, the fact that \( |f'(x)| \leq M \) for each \( x \in [ma, ma + \eta(b, a, m)] \), and taking the modulus, we have

\[
\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right| \\
- \frac{1}{\eta(b, a, m)} \left[ \int_{ma}^{ma+\eta(x,a,m)} f(u) du - \int_{mb}^{mb+\eta(x,b,m)} f(u) du \right] \\
\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left[ \int_0^1 (1-t)^p dt \right]^\frac{1}{p} \left( \int_0^1 |f'(ma + t\eta(x, a, m))|^q dt \right)^\frac{1}{q} \\
+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left[ \int_0^1 (1-t)^p dt \right]^\frac{1}{p} \left( \int_0^1 |f'(mb + t\eta(x, b, m))|^q dt \right)^\frac{1}{q} \\
\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left[ \int_0^1 (1-t)^p dt \right]^\frac{1}{p} \left[ \int_0^1 \left( \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^\frac{1}{q} \\
+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left[ \int_0^1 (1-t)^p dt \right]^\frac{1}{p} \left[ \int_0^1 \left( \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^\frac{1}{q} \\
\leq \frac{M}{(p+1)^{1/p}} \left( \frac{m\pi}{2} \right)^\frac{1}{q} \left[ \frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\]

\[\Box\]

Remark 6. In Theorem 6, if we choose \( m = 1 \) and \( \eta(x, y, 1) = x - y \) then we get (see [13], Theorem 2.4).

Theorem 7. Let \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invex subset with respect to \( \eta : A \times A \times (0, 1) \rightarrow \mathbb{R}_0 \) for any fixed \( m \in (0, 1] \) and let \( a, b \in A \), \( a < b \) with \( ma < ma + \eta(b, a, m) \). Assume that \( f : A \rightarrow \mathbb{R} \) is a differentiable function on \( A^o \). If \( |f'|^q \) is a \( MT_m \)-preinvex function on \( [ma, ma + \eta(b, a, m)] \), \( q \geq 1 \) and \( |f'(x)| \leq M \), then for each \( x \in [ma, ma + \eta(b, a, m)] \), we have

\[
\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right| \\
\leq \frac{M}{(p+1)^{1/p}} \left( \frac{m\pi}{2} \right)^\frac{1}{q} \left[ \frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].
\]
we get Theorem 5.

and taking the modulus, we have mean inequality, the fact that then we get (see [13], Theorem 2.6). Also, in Theorem 7, if we choose $q$

Using Lemma 2,

Proof. Using Lemma 2, $MT_m$-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\left| \frac{\eta(x, a, m)f(ma) - \eta(x, b, m)f(mb)}{\eta(b, a, m)} \right| \leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \int_0^1 |t - 1||f'(ma + t\eta(x, a, m))|dt$$

$$+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \int_0^1 |1 - t||f'(mb + t\eta(x, b, m))|dt$$

$$\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t)dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 (1 - t)|f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}}$$

$$+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t)dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 (1 - t)|f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x, a, m)^2}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t)dt \right)^{1 - \frac{1}{q}}$$

$$\times \left[ \int_0^1 (1 - t) \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} |f'(x)|^q + \frac{m\sqrt{1 - t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^\frac{1}{q}$$

$$+ \frac{\eta(x, b, m)^2}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t)dt \right)^{1 - \frac{1}{q}}$$

$$\times \left[ \int_0^1 (1 - t) \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} |f'(x)|^q + \frac{m\sqrt{1 - t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^\frac{1}{q}$$

$$\leq M \left( \frac{1}{2} \right)^{1 + \frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[ \frac{\eta(x, a, m)^2 + \eta(x, b, m)^2}{|\eta(b, a, m)|} \right].$$

Remark 7. In Theorem 7, if we choose $m = 1$ and $\eta(x, y, 1) = x - y$ then we get (see [13], Theorem 2.6). Also, in Theorem 7, if we choose $q = 1$, we get Theorem 5.
4. Hermite-Hadamard type fractional integral inequalities for $MT_m$-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for $MT_m$-preinvex functions via fractional integrals, we need the following new lemma:

**Lemma 3.** Let $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0,1] \rightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $a,b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \rightarrow \mathbb{R}$ is a differentiable function on $K^\circ$ and $f'$ is integrable on $[ma,ma + \eta(b,a,m)]$. Then, for each $x \in [ma,ma + \eta(b,a,m)]$ and $\alpha > 0$, we have

\begin{equation}
\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)
= \frac{\Gamma(\alpha + 1)}{\eta(b,a,m)} \left[ J_\alpha^{\eta(x,a,m)} f(ma) - J_\alpha^{\eta(x,b,m)} f(mb) \right]
- \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^{\alpha} - 1)f'(ma + t\eta(x,a,m))dt
+ \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^{\alpha})f'(mb + t\eta(x,b,m))dt,
\end{equation}

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler Gamma function.

**Proof.** Denote

\begin{align*}
I &= \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^{\alpha} - 1)f'(ma + t\eta(x,a,m))dt \\
&\quad + \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^{\alpha})f'(mb + t\eta(x,b,m))dt.
\end{align*}

Integrating by parts, we get

\begin{align*}
I &= \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \left[ (t^{\alpha} - 1)\frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)} \right]_0^1 \\
&\quad - \alpha \int_0^1 t^{\alpha-1}f(ma + t\eta(x,a,m)) \eta(x,a,m) dt \\
&\quad + \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \left[ (1 - t^{\alpha})\frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)} \right]_0^1 \\
&\quad + \alpha \int_0^1 t^{\alpha-1}f(mb + t\eta(x,b,m)) \eta(x,b,m) dt.
\end{align*}
\[ \frac{\eta(x,a,m)^\alpha f(ma) - \eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)} - \frac{\Gamma(\alpha + 1)}{\eta(b,a,m)} \left[ J_{(ma+\eta(x,a,m))}^\alpha f(ma) - J_{(mb+\eta(x,b,m))}^\alpha f(mb) \right]. \]

**Remark 8.** Clearly, if we choose \( m = 1 \) and \( \eta(x,y,1) = x - y \) in Lemma 3, we get (see [13], Lemma 3.1).

By using Lemma 3, one can extend to the following results.

**Theorem 8.** Let \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invex subset with respect to \( \eta : A \times A \times (0,1) \rightarrow \mathbb{R}_0 \) for any fixed \( m \in (0,1) \) and let \( a,b \in A, a < b \) with \( ma < ma + \eta(b,a,m) \). Assume that \( f : A \rightarrow \mathbb{R} \) is a differentiable function on \( A^\circ \). If \( |f'| \) is a \( MT_m \)-preinvex function on \( [ma, ma + \eta(b,a,m)] \) and \( |f'(x)| \leq M \), then for each \( x \in [ma, ma + \eta(b,a,m)] \) and \( \alpha > 0 \), we have

\[ Mm = \frac{1}{2} \left[ \frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \right] \left[ \frac{\pi}{\Gamma(\alpha + \frac{1}{2})} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \right]. \]

**Proof.** Using Lemma 3, \( MT_m \)-preinvexity of \( |f'| \), the fact that \( |f'(x)| \leq M \) for each \( x \in [ma, ma + \eta(b,a,m)] \), \( \alpha > 0 \), and taking the modulus, we have

\[ \left| \frac{\eta(x,a,m)^\alpha f(ma) - \eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)} - \frac{\Gamma(\alpha + 1)}{\eta(b,a,m)} \left[ J_{(ma+\eta(x,a,m))}^\alpha f(ma) - J_{(mb+\eta(x,b,m))}^\alpha f(mb) \right] \right| \]

\[ \leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 |t^\alpha - 1||f'(ma + t\eta(x,a,m))|dt \]

\[ + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 |1 - t^\alpha||f'(mb + t\eta(x,b,m))|dt \]

\[ \leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 (1 - t^\alpha) \left[ \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)| \right] dt \]

\[ + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 (1 - t^\alpha) \left[ \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right] dt \]
\[ \leq \frac{Mm}{2} \left[ |\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1} \right] \left[ \pi - \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha + 1)} \right]. \]

\[ \text{Remark 9.} \] In Theorem 8, if we choose \( m = 1 \) and \( \eta(x,y,1) = x-y \) then we get (see [13], Theorem 3.2). Also, in Theorem 8, if we choose \( \alpha = 1 \), we get the inequality in Theorem 5.

\[ \text{Theorem 9.} \] Let \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invel subset with respect to \( \eta: A \times A \times (0,1) \rightarrow \mathbb{R}_0 \) for any fixed \( m \in (0,1] \) and let \( a, b \in A \), \( a < b \) with \( ma < ma + \eta(b,a,m) \). Assume that \( f: A \rightarrow \mathbb{R} \) is a differentiable function on \( A^0 \). If \( |f'|^q \) is a \( MT_m \)-preinvex function on \([ma, ma + \eta(b,a,m)]\), \( q > 1 \), \( p^{-1} + q^{-1} = 1 \) and \( |f'(x)| \leq M \), then for each \( x \in [ma, ma + \eta(b,a,m)] \) and \( \alpha > 0 \), we have

\[ \left(11\right) \left| \frac{\eta(x,a,m)\alpha f(ma) - \eta(x,b,m)\alpha f(mb)}{\eta(b,a,m)} \right| \]

\[-\frac{\Gamma(\alpha + 1)}{\eta(b,a,m)} \left[ J_{(ma + \eta(x,a,m))}^{\alpha} (f(ma) - J_{(mb + \eta(x,b,m))}^{\alpha} f(mb)) \right] \]

\[ \leq M \left( \frac{m\pi}{2} \right)^{\frac{1}{q}} \left[ \frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{\eta(b,a,m)} \right] \left[ \frac{\Gamma(p + 1) \Gamma\left(\frac{1}{\alpha}\right)}{\alpha \Gamma(p + 1 + \frac{1}{\alpha})} \right]^{\frac{1}{p}}. \]

\[ \text{Proof.} \] Suppose that \( q > 1 \). Using Lemma 3, \( MT_m \)-preinvexity of \( |f'|^q \), Hölder inequality, the fact that \( |f'(x)| \leq M \) for each \( x \in [ma, ma + \eta(b,a,m)] \), \( \alpha > 0 \), and taking the modulus, we have

\[ \left| \frac{\eta(x,a,m)\alpha f(ma) - \eta(x,b,m)\alpha f(mb)}{\eta(b,a,m)} \right| \]

\[-\frac{\Gamma(\alpha + 1)}{\eta(b,a,m)} \left[ J_{(ma + \eta(x,a,m))}^{\alpha} (f(ma) - J_{(mb + \eta(x,b,m))}^{\alpha} f(mb)) \right] \]

\[ \leq \left| \frac{\eta(x,a,m)}{\eta(b,a,m)} \right|^{\alpha+1} \int_0^1 |t^\alpha - 1| |f'(ma + t\eta(x,a,m))|dt \]

\[ + \left| \frac{\eta(x,b,m)}{\eta(b,a,m)} \right|^{\alpha+1} \int_0^1 |1 - t^\alpha| |f'(mb + t\eta(x,b,m))|dt \]

\[ \leq \left| \frac{\eta(x,a,m)}{\eta(b,a,m)} \right|^{\alpha+1} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(ma + t\eta(x,a,m))|^q dt \right)^{\frac{1}{q}} \]

\[ + \left| \frac{\eta(x,b,m)}{\eta(b,a,m)} \right|^{\alpha+1} \left( \int_0^1 (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(mb + t\eta(x,b,m))|^q dt \right)^{\frac{1}{q}}. \]
\[ \leq \frac{|\eta(x, a, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha)^{\frac{1}{p}} dt \right) \]

\[ \times \left[ \int_0^1 \left( \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right] \]

\[ + \frac{|\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha)^{\frac{1}{p}} dt \right) \]

\[ \times \left[ \int_0^1 \left( \frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right] \]

\[ \leq M \left(\frac{m\pi}{2}\right)^{\frac{1}{q}} \left[ \frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \Gamma(p+1) \left(\frac{1}{\alpha}\right) \right]^{\frac{1}{p}}. \]

**Remark 10.** In Theorem 9, if we choose \( m = 1 \) and \( \eta(x, y, 1) = x - y \) then we get (see [13], Theorem 3.5). Also, in Theorem 9, if we choose \( \alpha = 1 \), we get the inequality in Theorem 6.

**Theorem 10.** Let \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invex subset with respect to \( \eta : A \times A \times (0, 1) \rightarrow \mathbb{R}_0 \) for any fixed \( m \in (0, 1) \) and let \( a, b \in A \), \( a < b \) with \( ma < ma + \eta(b, a, m) \). Assume that \( f : A \rightarrow \mathbb{R} \) is a differentiable function on \( A^\circ \). If \( |f|^q \) is a \( MT_m \)-preinvex function on \( [ma, ma + \eta(b, a, m)] \), \( q \geq 1 \) and \( |f'(x)| \leq M \), then for each \( x \in [ma, ma + \eta(b, a, m)] \) and \( \alpha > 0 \), we have

\[ |\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)| \]

\[ \leq M \left(\frac{m\pi}{2}\right)^{\frac{1}{q}} \left[ \frac{\alpha}{\alpha + 1} \right]^{1 - \frac{1}{q}} \left[ \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right]^{\frac{1}{p}} \]

\[ \times \left[ \frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right]. \]

**Proof.** Using Lemma 3, \( MT_m \)-preinvexity of \( |f|^q \), the well-known power mean inequality, the fact that \( |f'(x)| \leq M \) for each \( x \in [ma, ma + \eta(b, a, m)] \), \( \alpha > 0 \), and taking the modulus, we have

\[ |\eta(x, a, m)^\alpha f(ma) - \eta(x, b, m)^\alpha f(mb)| \]

\[ \leq M \left(\frac{m\pi}{2}\right)^{\frac{1}{q}} \left[ \frac{\alpha}{\alpha + 1} \right]^{1 - \frac{1}{q}} \left[ \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right]^{\frac{1}{p}} \]

\[ \times \left[ \frac{|\eta(x, a, m)|^{\alpha+1} + |\eta(x, b, m)|^{\alpha+1}}{|\eta(b, a, m)|} \right]. \]
\[
\begin{align*}
&\quad - \frac{\Gamma(\alpha + 1)}{\eta(b, a, m)} \left[ J_\eta^{\alpha}(ma + \eta(x, a, m)) - f(ma) - J_\eta^{\alpha}(mb + \eta(x, b, m)) - f(mb) \right] \\
&\;\leq \frac{\eta(x, a, m)^{\alpha + 1}}{|\eta(b, a, m)|} \int_0^1 |t^\alpha - 1| |f'(ma + t\eta(x, a, m))| dt \\
&\quad + \frac{\eta(x, b, m)^{\alpha + 1}}{|\eta(b, a, m)|} \int_0^1 |1 - t^\alpha| |f'(mb + t\eta(x, b, m))| dt \\
&\;\leq \frac{\eta(x, a, m)^{\alpha + 1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \\
&\quad \times \left( \int_0^1 (1 - t^\alpha)|f'(ma + t\eta(x, a, m))|^q dt \right)^{\frac{1}{q}} \\
&\quad + \frac{\eta(x, b, m)^{\alpha + 1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \\
&\quad \times \left( \int_0^1 (1 - t^\alpha)|f'(mb + t\eta(x, b, m))|^q dt \right)^{\frac{1}{q}} \\
&\;\leq \frac{\eta(x, a, m)^{\alpha + 1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \\
&\quad \times \left[ \int_0^1 (1 - t^\alpha) \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} |f'(x)|^q + \frac{m\sqrt{1 - t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
&\quad + \frac{\eta(x, b, m)^{\alpha + 1}}{|\eta(b, a, m)|} \left( \int_0^1 (1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \\
&\quad \times \left[ \int_0^1 (1 - t^\alpha) \left( \frac{m\sqrt{t}}{2\sqrt{1 - t}} |f'(x)|^q + \frac{m\sqrt{1 - t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
&\quad \leq M \left( \frac{m}{2} \right)^{\frac{1}{q}} \left( \frac{\alpha}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left[ \frac{\pi}{\Gamma(\alpha + \frac{1}{2})} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \right]^{\frac{1}{q}} \\
&\quad \times \left[ \frac{\eta(x, a, m)^{\alpha + 1} + |\eta(x, b, m)|^{\alpha + 1}}{|\eta(b, a, m)|^{\alpha + 1}} \right].
\end{align*}
\]

Remark 11. In Theorem 10, if we choose \( m = 1 \) and \( \eta(x, y, m) = x - my \) then we get (see [13], Theorem 3.8). Also, in Theorem 10, if we choose \( \alpha = 1 \), we get Theorem 7.

5. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.
Definition 9 (see [2]). A function $M : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$.
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers $\alpha, \beta$ ($\alpha \neq \beta$).

1. The arithmetic mean:
   $$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:
   $$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$  

3. The harmonic mean:
   $$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$  

4. The power mean:
   $$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:
   $$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$  

6. The logarithmic mean:
   $$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\log(\beta) - \log(\alpha)}, \quad \alpha \neq \beta.$$  

7. The generalized log-mean:
   $$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha \neq \beta.$$  

8. The weighted $p$-power mean:
   $$M_p\left(\frac{\alpha_1}{u_1}, \frac{\alpha_2}{u_2}, \cdots, \frac{\alpha_n}{u_n}\right) = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$
   where $0 \leq \alpha_i \leq 1$, $u_i > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^{n} \alpha_i = 1$. 
It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let $a$ and $b$ be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \rightarrow \mathbb{R}_+$, which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Sections (- ) for these means as follows:

Replace $\eta(x, y, m)$ with $\eta(x, y)$ and setting $\eta(a, x) = M(a, x)$ and $\eta(b, x) = M(b, x)$, $\forall x \in A$, for value $m = 1$ in (7), (8), (11) and (12) one can obtain the following interesting inequalities involving means:

\[
\begin{align*}
(13) & \quad \left| \frac{\eta(a, x)f(a) - \eta(b, x)f(b)}{M(a, b)} \right| \\
& - \frac{1}{M(a, b)} \left[ \int_a^{a+M(a, x)} f(u)du - \int_b^{b+M(b, x)} f(u)du \right] \\
& \leq \frac{M}{(p + 1)^{1/p}} \left( \frac{\pi}{2} \right)^\frac{1}{q} \left[ \frac{M(a, x)^2 + M(b, x)^2}{M(a, b)} \right],
\end{align*}
\]

\[
\begin{align*}
(14) & \quad \left| \frac{\eta(a, x)f(a) - \eta(b, x)f(b)}{M(a, b)} \right| \\
& - \frac{1}{M(a, b)} \left[ \int_a^{a+M(a, x)} f(u)du - \int_b^{b+M(b, x)} f(u)du \right] \\
& \leq M \left( \frac{1}{2} \right)^{1+\frac{1}{q}} \left( \frac{\pi}{2} \right)^\frac{1}{q} \left[ \frac{M(a, x)^2 + M(b, x)^2}{M(a, b)} \right],
\end{align*}
\]

\[
\begin{align*}
(15) & \quad \left| \frac{M(a, x)^\alpha f(a) - M(b, x)^\alpha f(b)}{M(a, b)} \right| \\
& - \frac{\Gamma(\alpha + 1)}{M(a, b)} \left[ J^\alpha_{(a+M(a, x))} - f(a) - J^\alpha_{(b+M(b, x))} - f(b) \right] \\
& \leq M \left( \frac{\pi}{2} \right)^\frac{1}{q} \left[ \frac{M(a, x)^{\alpha+1} + M(b, x)^{\alpha+1}}{M(a, b)} \right] \left[ \frac{\Gamma(p + 1)\Gamma \left( \frac{1}{\alpha} \right)}{\alpha\Gamma(p + 1 + \frac{1}{\alpha})} \right]^{\frac{1}{p}},
\end{align*}
\]

\[
\begin{align*}
(16) & \quad \left| \frac{M(a, x)^\alpha f(a) - M(b, x)^\alpha f(b)}{M(a, b)} \right| \\
& - \frac{\Gamma(\alpha + 1)}{M(a, b)} \left[ J^\alpha_{(a+M(a, x))} - f(a) - J^\alpha_{(b+M(b, x))} - f(b) \right]
\end{align*}
\]
Letting \( M(a, x) \) and \( M(b, x) \) equal to \( A, G, H, P_r, I, L, L_p, M_p, \forall x \in A \) in (13), (14), (15) and (16), we get the inequalities involving means for a particular choices of a differentiable \( MT_1 \)-preinvex function \( f \). The details are left to the interested reader.

References

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