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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GENERALIZED $(r; g, s, m, \varphi)$-PREINVEX FUNCTIONS

ABSTRACT. In the present paper, a new class of generalized $(r; g, s, m, \varphi)$-preinvex functions is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving generalized $(r; g, s, m, \varphi)$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are established. These results not only extend the results appeared in the literature (see [1],[2]), but also provide new estimates on these types.

KEY WORDS: Hermite-Hadamard type inequality, Hölder’s inequality, Minkowski’s inequality, Cauchy’s inequality, power mean inequality, Riemann-Liouville fractional integral, $s$-convex function in the second sense, $m$-invex, $P$-function.

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1. Introduction and preliminaries

The following notation is used throughout this paper. We use $I$ to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and $I^o$ to denote the interior of $I$. For any subset $K \subseteq \mathbb{R}^n$, $K^o$ is used to denote the interior of $K$. $\mathbb{R}^n$ is used to denote a $n$-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.** Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on $I$ and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$
Fractional calculus (see [14]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

**Definition 1.** Let \( f \in L^1_{[a,b]} \). The Riemann-Liouville integrals \( J^\alpha_{a+}f \) and \( J^\alpha_{b-}f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a
\]

and

\[
J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,
\]

where \( \Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du \). Here \( J^0_{a+}f(x) = J^0_{b-}f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13],[14]).

Now, let us recall some definitions of various convex functions.

**Definition 2** (see [4]). A nonnegative function \( f : I \subset \mathbb{R} \rightarrow [0, +\infty) \) is said to be \( P \)-function or \( P \)-convex, if

\[
f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \; t \in [0,1].
\]

**Definition 3** (see [5]). A function \( f : [0, +\infty) \rightarrow \mathbb{R} \) is said to be \( s \)-convex in the second sense, if

\[
(2) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)
\]

for all \( x, y \geq 0, \; \lambda \in [0,1] \) and \( s \in (0,1] \).

It is clear that a 1-convex function must be convex on \([0, +\infty)\) as usual. The \( s \)-convex functions in the second sense have been investigated in (see [5]).

**Definition 4** (see [6]). A set \( K \subset \mathbb{R}^n \) is said to be invex with respect to the mapping \( \eta : K \times K \rightarrow \mathbb{R}^n \), if \( x + t\eta(y,x) \in K \) for every \( x, y \in K \) and \( t \in [0,1] \).

Notice that every convex set is invex with respect to the mapping \( \eta(y, x) = y - x \), but the converse is not necessarily true. For more details (see [6],[7]).
Definition 5 (see [8]). The function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to $\eta$, if for every $x, y \in K$ and $t \in [0, 1]$, we have that
\[
f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).
\]
The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following form:
\[
\int_a^b (x - a)^p(b - x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^*|f|,
\]
for certain $B_{m,k}, \gamma_k$ and rest $R_m^*|f|$ (see [9]). Recently, Liu (see [10]) obtained several integral inequalities for the left hand side of (3) under the Definition 2 of $P$-function. Also in (see [11]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized $(r; g, s, m, \varphi)$-preinvex function is introduced and some new integral inequalities for the left hand side of (3) involving generalized $(r; g, s, m, \varphi)$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions via fractional integrals are given. These general inequalities give us some new estimates for the left hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for generalized $(r; g, s, m, \varphi)$-preinvex functions

Definition 6 (see [3]). A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$.

Definition 7 (see [12]). A positive function $f$ on the invex set $K$ is said to be logarithmically preinvex, if
\[
f(u + t\eta(v, u)) \leq f^{1-t}(u)f^t(v)
\]
for all $u, v \in K$ and $t \in [0, 1]$. 
\textbf{Definition 8} (see [12]). The function \( f \) on the invex set \( K \) is said to be \( r \)-preinvex with respect to \( \eta \), if
\[
f(u + t\eta(v, u)) \leq M_r(f(u), f(v); t)
\]
holds for all \( u, v \in K \) and \( t \in [0, 1] \), where
\[
M_r(x, y; t) = \begin{cases}
(1-t)x^r + ty^r \, & \text{if } r \neq 0 \\
(1-t)x \, & \text{if } r = 0,
\end{cases}
\]
is the weighted power mean of order \( r \) for positive numbers \( x \) and \( y \).

We next give new definition, to be referred as generalized \((r; g, s, m, \varphi)\)-preinvex function.

\textbf{Definition 9.} Let \( K \subseteq \mathbb{R} \) be an open \( m \)-invex set with respect to \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \), \( g : [0, 1] \rightarrow [0, 1] \) be a differentiable function and \( \varphi : I \rightarrow K \) is a continuous function. The function \( f : K \rightarrow (0, +\infty) \) is said to be generalized \((r; g, s, m, \varphi)\)-preinvex with respect to \( \eta \), if
\[
f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \leq M_r(f(\varphi(x)), f(\varphi(y)), m; s; t)
\]
holds for any fixed \( s, m \in (0, 1] \) and for all \( x, y \in I, t \in [0, 1] \), where
\[
M_r(f(\varphi(x)), f(\varphi(y)), m; s; t) = \begin{cases}
\left[ m(1-g(t))^r f^r(\varphi(x)) + g^r(t) f^r(\varphi(y)) \right]^{\frac{1}{r}} \, & \text{if } r \neq 0 \\
\left[ f(\varphi(x)) \right]^m (1-g(t))^s \left[ f(\varphi(y)) \right]^g(t), \, & \text{if } r = 0,
\end{cases}
\]
is the weighted power mean of order \( r \) for positive numbers \( f(\varphi(x)) \) and \( f(\varphi(y)) \).

\textbf{Remark 2.} In Definition 9, it is worthwhile to note that the class of generalized \((r; g, s, m, \varphi)\)-preinvex function is a generalization of the class of \( s \)-convex in the second sense function given in Definition 3. Also, for \( r = 1, g(t) = t, \forall t \in [0, 1] \) and \( \varphi(x) = x, \forall x \in I \), we get the notion of generalized \((s, m)\)-preinvex function (see [3]).

\textbf{Example 1.} Let \( f(x) = -|x|, g(t) = t, \varphi(x) = x, r = s = 1 \) and
\[
\eta(y, x, m) = \begin{cases}
y - mx, & \text{if } x \geq 0, \ y \geq 0 \\
y - mx, & \text{if } x \leq 0, \ y \leq 0 \\
x - mx, & \text{if } x \geq 0, \ y \leq 0 \\
x - mx, & \text{if } x \leq 0, \ y \geq 0.
\end{cases}
\]
Then \( f(x) \) is a generalized \((1; t, 1, m, x)\)-preinvex function of with respect to \( \eta : \mathbb{R} \times \mathbb{R} \times (0, 1) \rightarrow \mathbb{R} \) and any fixed \( m \in (0, 1] \). However, it is obvious that \( f(x) = -|x| \) is not a convex function on \( \mathbb{R} \).
In this section, in order to prove our main results regarding some new integral inequalities involving generalized \((r; g, s, m, \varphi)\)-preinvex functions, we need the following new interesting lemma:

**Lemma 1.** Let \(\varphi : I \rightarrow K\) be a continuous function and \(g : [0, 1] \rightarrow [0, 1]\) is a differentiable function. Assume that \(f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow \mathbb{R}\) is a continuous function on \(K^\circ\) with respect to \(\eta : K \times K \times (0, 1) \rightarrow \mathbb{R}\), for \(m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)\). Then for any fixed \(m \in (0, 1)\) and \(p, q > 0\), we have

\[
\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\
\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].
\]

**Proof.** It is easy to observe that

\[
\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\
\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q \\
\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\
\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].
\]

\[\blacksquare\]

**Theorem 2.** Let \(\varphi : I \rightarrow K\) be a continuous function and \(g : [0, 1] \rightarrow [0, 1]\) is a differentiable function. Assume that \(f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)\) is a continuous function on \(K^\circ\) with \(m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)\). Let \(k > 1\) and \(0 < r \leq 1\). If \(f^{k-1}\) is generalized \((r; g, s, m, \varphi)\)-preinvex function on an open \(m\)-invex set \(K\) with respect to \(\eta : K \times K \times (0, 1) \rightarrow \mathbb{R}\) for any fixed \(s, m \in (0, 1]\), then for any fixed \(p, q > 0\),

\[
\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) \\
+ \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left(\frac{r}{s+r}\right)^{\frac{k-1}{k}} B_1^{\frac{1}{k}} (g(t); k, p, q)
\]
\[
\times \left[ m \left( (1 - g(0))^{\frac{s}{r}} + 1 - (1 - g(1))^{\frac{s}{r}} + 1 \right)^{\frac{1}{k}} f^{\frac{r}{k}}(\varphi(a)) \\
+ \left( g^{\frac{s}{r}} + 1 (1 - g^{\frac{s}{r}} + 1(0)) \right)^{\frac{1}{k}} f^{\frac{r}{k}}(\varphi(b)) \right]^{\frac{k-1}{k}},
\]

where \( B(g(t); k, p, q) = \int_{0}^{1} g^{kp}(t)(1 - g(t))^{kq}d[g(t)]. \)

**Proof.** Let \( k > 1 \) and \( 0 < r \leq 1 \). Since \( f^{\frac{k}{k-r}} \) is generalized \((r; g, s, m, \varphi)\)-preinvex function on \( K \), combining with Lemma 1, Hölder inequality and Minkowski inequality for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1) \), we get

\[
\int_{m \varphi(a)}^{m \varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m \varphi(a))^{p}(m \varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^{q}f(x)dx
\]

\[
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[ \int_{0}^{1} g^{kp}(t)(1 - g(t))^{kq}d[g(t)] \right]^{\frac{1}{k}}
\]

\[
\times \left[ \int_{0}^{1} |f(m \varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))|^{\frac{k}{k-r}}d[g(t)] \right]^{\frac{k-1}{k-r}}
\]

\[
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q)
\]

\[
\times \left[ \int_{0}^{1} \left( m^{s} f^{\frac{k}{k-r}}(\varphi(a)) + g^{s}(t) f^{\frac{k}{k-r}}(\varphi(b)) \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{k-1}{k-r}}
\]

\[
= |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left( \frac{r}{s+r} \right)^{\frac{k-1}{k}} B^{\frac{1}{k}}(g(t); k, p, q)
\]

\[
\times \left[ m \left( (1 - g(0))^{\frac{s}{r}} + 1 - (1 - g(1))^{\frac{s}{r}} + 1 \right)^{\frac{1}{k}} f^{\frac{r}{k}}(\varphi(a)) \\
+ \left( g^{\frac{s}{r}} + 1 (1 - g^{\frac{s}{r}} + 1(0)) \right)^{\frac{1}{k}} f^{\frac{r}{k}}(\varphi(b)) \right]^{\frac{k-1}{k-r}}.
\]
Corollary 1. Under the same conditions as in Theorem 2 for \( r = 1 \) and 
\( g(t) = t \), we get (see [1], Theorem 2.2).

Theorem 3. Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Assume that \( f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty) \) is a continuous function on \( K^\circ \) with \( m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m) \). Let \( l \geq 1 \) and \( 0 < r \leq 1 \). If \( f^l \) is generalized \((r; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K \) with respect to 
\( \eta : K \times K \times (0, 1] \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1] \), then for any fixed 
\( p, q > 0 \),

\[
\begin{align*}
\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{r-1}{r}} (g(t); p, q) \\
\times \left[ m f^l(\varphi(a)) B^r (g(t); p, q + \frac{s}{r}) + f^l(\varphi(b)) B^r (g(t); p + \frac{s}{r}, q) \right]^{\frac{1}{r}},
\end{align*}
\]

where \( B(g(t); p, q) = \int_0^1 g^p(t)(1 - g(t))^q d[g(t)] \).

Proof. Let \( l \geq 1 \) and \( 0 < r \leq 1 \). Since \( f^l \) is generalized \((r; g, s, m, \varphi)\)-preinvex function on \( K \), combining with Lemma 1, the well-known power mean inequality and Minkowski inequality for all \( t \in [0, 1] \) and for any fixed 
\( s, m \in (0, 1] \), we get

\[
\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
= |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \\
\times \int_0^1 \left[ g^p(t)(1 - g(t))^q \right]^{\frac{r-1}{r}} \left[ g^p(t)(1 - g(t))^q \right]^{\frac{1}{r}} \\
\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)] \\
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[ \int_0^1 g^p(t)(1 - g(t))^q d[g(t)] \right]^{\frac{r-1}{r}} \\
\times \left[ \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)] \right]^{\frac{1}{r}} \\
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{r-1}{r}} (g(t); p, q) \\
\times \left[ \int_0^1 g^p(t)(1 - g(t))^q \left( m(1 - g(t))^s f^l(\varphi(a)) \right) \right].
\]
\[+ g^s(t)f^r_l(\varphi(b)) \frac{1}{r} d[g(t)] \right]^\frac{1}{r}\]
\[\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{-r}}(g(t); p, q) \]
\[\times \left[ \left( \int_0^1 m^\frac{1}{r} g^p(t) (1 - g(t))^{q+\frac{z}{r}} f^l(\varphi(a)) d[g(t)] \right)^r \right.\]
\[+ \left. \left( \int_0^1 g^{p+\frac{z}{r}}(t) (1 - g(t))^q f^l(\varphi(b)) d[g(t)] \right)^r \right]^{\frac{1}{r}} \]
\[= |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{-r}}(g(t); p, q) \]
\[\times \left[ m f^l(\varphi(a)) B^r (g(t); p, q + \frac{s}{r}) + f^l(\varphi(b)) B^r (g(t); p + \frac{s}{r}, q) \right]^{\frac{1}{r}} .\]

\[\text{Corollary 2. Under the same conditions as in Theorem 3 for } r = 1 \text{ and } g(t) = t, \text{ we get (see [1], Theorem 2.3).}\]

3. Hermite-Hadamard type fractional integral inequalities for generalized \((r; g, s, m, \varphi)\)-preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized \((r; g, s, m, \varphi)\)-preinvex functions via fractional integrals.

**Theorem 4.** Let \(\varphi : I \rightarrow K\) be a continuous function and \(g : [0, 1] \rightarrow [0, 1]\) is a differentiable function. Suppose \(K \subseteq \mathbb{R}\) be an open \(m\)-invex subset with respect to \(\eta : K \times K \times (0, 1) \rightarrow \mathbb{R}\) for any fixed \(s, m \in (0, 1)\) with \(m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)\). Assume that \(f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)\) be generalized \((r; g, s, m, \varphi)\)-preinvex function on an open \(m\)-invex set \(K^\circ\). Then for \(\alpha > 0\) and \(0 < r \leq 1\), we have

\[\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a) + g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a) + g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \]
\[\leq \left[ m f^l(\varphi(a)) B^r (g(t); \alpha - 1, \frac{s}{r}) \right.\]
\[+ \left. f^r(\varphi(b)) \left( \frac{r}{\alpha r + s} \right)^r \left( g^{\frac{z}{r} + \alpha}(1) - g^{\frac{z}{r} + \alpha}(0) \right) \right]^{\frac{1}{r}} .\]
Proof. Let $0 < r \leq 1$. Since $f$ is generalized $(r; g, s, m, \varphi)$-preinvex function on an open $m$-invex set $K^\circ$, combining with Minkowski inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

\[
\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a) + g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a) + g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha - 1} f(t) dt
\]

\[
= \int_{0}^{1} g^{\alpha - 1}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]
\]

\[
\leq \int_{0}^{1} g^{\alpha - 1}(t) \left[ m(1 - g(t))^s f^r(\varphi(a)) + g^s(t) f^r(\varphi(b)) \right]^{\frac{1}{r}} d[g(t)]
\]

\[
\leq \left\{ \left[ \int_{0}^{1} g^{\alpha - 1 + \frac{s}{r}}(t) f(\varphi(b)) d[g(t)] \right]^{\frac{1}{r}} + \left[ \int_{0}^{1} m^{\frac{1}{r}} g^{\alpha - 1}(t)(1 - g(t))^\frac{s}{r} f(\varphi(a)) d[g(t)] \right]^{\frac{1}{r}} \right\}
\]

\[
= \left[ m f^r(\varphi(a)) B^r \left( g(t); \alpha - 1, \frac{s}{r} \right) + f^r(\varphi(b)) \left( \frac{r}{\alpha r + s} \right)^{\frac{r}{r}} \left( g^{\frac{s}{r} + \alpha}(1) - g^{\frac{s}{r} + \alpha}(0) \right)^{\frac{1}{r}} \right].
\]

\[
\blacksquare
\]

Corollary 3. Under the same conditions as in Theorem 4 for $m = s = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.1).

Theorem 5. Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0, 1) \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1)$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty)$ are respectively generalized $(r; g, s, m, \varphi)$-preinvex function and generalized $(l; g, s, m, \varphi)$-preinvex function on an open $m$-invex set $K^\circ$. Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have

\[
\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a) + g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a) + g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha - 1} f(t) h(t) dt
\]

\[
\leq \frac{1}{2} \left\{ m f^r(\varphi(a)) B^r \left( g(t); \frac{2(\alpha - 1)}{r}, \frac{2s}{r} \right) + f^r(\varphi(b)) \left( \frac{s}{\alpha r + s} \right)^{\frac{1}{r}} \left( g^{\frac{s}{r} + \alpha}(1) - g^{\frac{s}{r} + \alpha}(0) \right)^{\frac{1}{r}} \right\}.
\]
\[+ f^r(\varphi(b)) \left( \frac{r}{2(\alpha - 1 + s) + r} \right)^\frac{r}{r} \left( g^\frac{2(\alpha-1+s)}{r} + 1 (1) - g^\frac{2(\alpha-1+s)}{r} + 1 (0) \right)^\frac{r}{r} \]
\[+ mh^l(\varphi(a)) B^l \left( g(t); \frac{2(\alpha-1)}{l}, \frac{2s}{l} \right)^\frac{l}{l} \]
\[+ h^l(\varphi(b)) \left( \frac{l}{2(\alpha - 1 + s) + l} \right)^\frac{l}{l} \left( g^\frac{2(\alpha-1+s)}{l} + 1 (1) - g^\frac{2(\alpha-1+s)}{l} + 1 (0) \right)^\frac{l}{l} \} \}

Proof. Let \( r > 1 \) and \( r^{-1} + l^{-1} = 1 \). Since \( f \) and \( h \) are respectively generalized \( (r; g, s, m, \varphi) \)-preinvex function and generalized \( (l; g, s, m, \varphi) \)-preinvex function on an open \( m \)-invex set \( K^o \), combining with Cauchy and Minkowski inequalities for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1] \), we get

\[
\frac{1}{\eta^a(\varphi(b), \varphi(a), m)} \int_{m \varphi(a) + g(0) \eta(\varphi(b), \varphi(a), m)}^{m \varphi(a) + g(1) \eta(\varphi(b), \varphi(a), m)} (t - m \varphi(a))^{a-1} f(t) h(t) dt
\]

\[
= \int_0^1 g^{(\alpha-1)}(\frac{t}{r} + \frac{s}{r})(t) \left( m \varphi(a) + g(t) \eta(\varphi(b), \varphi(a), m) \right)
\times h(m \varphi(a) + g(t) \eta(\varphi(b), \varphi(a), m)) d[g(t)]
\]

\[
\leq \int_0^1 g^{(\alpha-1)}(\frac{t}{r} + \frac{s}{r})(t) \left[ m(1 - g(t))^s f^r(\varphi(a)) + g^s(t) f^r(\varphi(b)) \right]^\frac{1}{r}
\times \left[ m(1 - g(t))^s h^l(\varphi(a)) + g^s(t) h^l(\varphi(b)) \right]^\frac{1}{l} d[g(t)]
\]

\[
\leq \frac{1}{2} \left\{ \int_0^1 \left[ g^{\alpha-1+s}(t) f^r(\varphi(b)) + mg^{\alpha-1}(t)(1 - g(t))^s f^r(\varphi(a)) \right]^\frac{2}{r} d[g(t)]
+ \int_0^1 \left[ g^{\alpha-1+s}(t) h^l(\varphi(b)) + mg^{\alpha-1}(t)(1 - g(t))^s h^l(\varphi(a)) \right]^\frac{2}{l} d[g(t)] \right\}
\]

\[
\leq \frac{1}{2} \left\{ \left( \int_0^1 g^{\frac{2(\alpha-1+s)}{r}}(t) f^2(\varphi(b)) d[g(t)] \right)^\frac{r}{r}
+ \left( \int_0^1 m^2 g^{\frac{2(\alpha-1)}{r}}(t)(1 - g(t))^\frac{2s}{r} f^2(\varphi(a)) d[g(t)] \right)^\frac{r}{r}
+ \left( \int_0^1 g^{\frac{2(\alpha-1+s)}{l}}(t) h^2(\varphi(b)) d[g(t)] \right)^\frac{l}{l}
+ \left( \int_0^1 m^2 g^{\frac{2(\alpha-1)}{l}}(t)(1 - g(t))^\frac{2s}{l} h^2(\varphi(a)) d[g(t)] \right)^\frac{l}{l} \right\} \]
\[\begin{align*}
= \frac{1}{2} \left\{ m f^r(\varphi(a))B^\frac{r}{2} \left( g(t); \frac{2(\alpha-1)}{r}, \frac{2s}{r} \right) \\
+ f^r(\varphi(b)) \left( \frac{r}{2(\alpha-1+s)+r} \right) ^\frac{r}{2} \left( g \left( \frac{2(\alpha-1+s)}{r}+1 \right) - g \left( \frac{2(\alpha-1+s)}{r}+1 \right) \right) ^\frac{1}{2} \\
+ mh^l(\varphi(a))B^\frac{l}{2} \left( g(t); \frac{2(\alpha-1)}{l}, \frac{2s}{l} \right) \\
+ h^l(\varphi(b)) \left( \frac{l}{2(\alpha-1+s)+l} \right) ^\frac{l}{2} \left( g \left( \frac{2(\alpha-1+s)}{l}+1 \right) - g \left( \frac{2(\alpha-1+s)}{l}+1 \right) \right) ^\frac{1}{2} \right\}. 
\end{align*}\]

**Corollary 4.** Under the same conditions as in Theorem 5 for \( m = s = 1 \), \( \varphi(x) = x \), \( \eta(\varphi(b), \varphi(a), m) = \eta(b, a) \) and \( g(t) = t \), we get (see [2], Theorem 3.3).

**Theorem 6.** Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Suppose \( K \subset \mathbb{R} \) be an open \( m \)-invex subset with respect to \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1] \) with \( m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m) \). Assume that \( f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty) \) are respectively generalized \((r; g, s, m, \varphi)\)-preinvex function and generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m \)-invex set \( K^o \). Then for \( \alpha > 0 \), \( r > 1 \) and \( r^{-1} + l^{-1} = 1 \), we have

\[
(9) \quad \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt \\
\leq \left\{ \frac{f^r(\varphi(b))}{s + \alpha} \left( g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + m f^r(\varphi(a))B(g(t); \alpha - 1, s) \right\} ^\frac{1}{r} \\
+ \left\{ \frac{h^l(\varphi(b))}{s + \alpha} \left( g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + m h^l(\varphi(a))B(g(t); \alpha - 1, s) \right\} ^\frac{1}{l}.
\]

**Proof.** Let \( r > 1 \) and \( r^{-1} + l^{-1} = 1 \). Since \( f \) and \( h \) are respectively generalized \((r; g, s, m, \varphi)\)-preinvex function and generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m \)-invex set \( K^o \), combining with Hölder inequality for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1] \), we get

\[
\frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t)h(t)dt
\]
\[
\begin{aligned}
&= \int_0^1 g^{(a-1)}(\frac{1}{r} + \frac{1}{l})(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
&\times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)] \\
\leq & \left\{ \int_0^1 \left[ g^{a-1+s}(t)f^r(\varphi(b)) + mg^{a-1}(t)(1 - g(t))^s f^r(\varphi(a)) \right] \right. \\
&\times \left( g^{a-1+s}(t)h^l(\varphi(b)) + mg^{a-1}(t)(1 - g(t))^s h^l(\varphi(a)) \right) \right\}^{\frac{1}{r}} \\
&+ \left\{ \int_0^1 \left[ g^{a-1+s}(t)h^l(\varphi(b)) + mg^{a-1}(t)(1 - g(t))^s h^l(\varphi(a)) \right] \right\}^{\frac{1}{t}} \\
&= \left\{ \frac{f^r(\varphi(b))}{s + \alpha} \left( g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + m f^r(\varphi(a)) B(g(t); \alpha - 1, s) \right\}^{\frac{1}{r}} \\
&+ \left\{ \frac{h^l(\varphi(b))}{s + \alpha} \left( g^{s+\alpha}(1) - g^{s+\alpha}(0) \right) + mh^l(\varphi(a)) B(g(t); \alpha - 1, s) \right\}^{\frac{1}{t}}.
\end{aligned}
\]

**Corollary 5.** Under the same conditions as in Theorem 6 for \( m = s = 1, \varphi(x) = x, \eta(\varphi(b), \varphi(a), m) = \eta(b, a) \) and \( g(t) = t \), we get (see [2], Theorem 3.9).

**Remark 3.** For different choices of positive values \( r, l = \frac{1}{2}, \frac{1}{3}, 2 \), etc., for any fixed \( s, m \in (0, 1) \), for a particular choices of a differentiable function \( g(t) = e^{-t}, \ln(t + 1), \sin(\frac{\pi t}{2}), \cos(\frac{\pi t}{2}) \), etc, and a particular choices of a continuous function \( \varphi(x) = e^x \) for all \( x \in \mathbb{R} \), \( x^n \) for all \( x > 0 \) and for all \( n \in \mathbb{N} \), etc, by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional inequalities.

**References**


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