CONTRA $(\mu g, \lambda)$-CONTINUOUS FUNCTIONS

Abstract. In this paper we introduce and study some properties of contra $(\mu g, \lambda)$-continuous functions. We obtain some characterizations and several properties of such functions.

Key words: contra $(\mu g, \lambda)$-continuous, $\mu g$-closed set, generalized topological space

AMS Mathematics Subject Classification: 54C10, 54C08, 54D10.

1. Introduction

In 2002, generalized topological spaces introduced and developed by A. Csaszar [1]. A generalized topology (briefly a GT) $\mu$ on a nonempty set $X$ is a collection of subsets of $X$ such that $\emptyset \in \mu$ and $\mu$ is closed under arbitrary unions. The pair $(X, \mu)$ is called a generalized topological space (briefly a GTS). Elements of $\mu$ are called $\mu$-open sets and a complement of a $\mu$-open set is called a $\mu$-closed set. The union of all $\mu$-open subsets of a subset $S$ of $(X, \mu)$ is called the $\mu$-interior of $S$ [2] and denoted by $i_{\mu}(S)$. The intersection of $\mu$-closed sets containing $S$ is called the $\mu$-closure of $S$ [2] and denoted by $c_{\mu}(S)$. A subset $S$ of a space $(X, \mu)$ is called $\mu$-regular closed (shortly $\mu r$-closed) [3] if $S = c_{\mu}(i_{\mu}(S))$. If $X \setminus S$ is $\mu$-regular closed then $S$ is called as $\mu$-regular open (shortly $\mu r$-open).

A GTS $(X, \mu)$ is called strong if $X \in \mu$ and a quasi-topological space if $\mu$ is closed under finite intersections. $(X, \mu)$ is said to be extremally disconnected (briefly EDC) if the $\mu$-closure of every $\mu$-open set is $\mu$-open.

Generalized closed sets introduced by N. Levine [6] in 1970. This notion has been studied and developed in many papers and plays a significant role in General Topology. The purpose of this paper is to introduce new types of continuous functions using this concept.

2. Preliminaries

Definition 1. A subset $A$ of a GTS $(X, \mu)$ is said to be $\mu$-semi-open [1] (respectively $\mu$-preopen [1], and $\mu$-$\delta$-open) if $A \subset c_{\mu}(i_{\mu}(A))$ (respectively
$A \subset i_{\mu}(c_{\mu}(A))$, and $A$ is the union of $\mu r$-open sets). The complements of the above sets are called respective closed ones.

**Definition 2.** Let $A$ be a subset of GTS $(X, \mu)$. Then, $A$ is called $\mu g$-closed [9] if $c_{\mu}(A) \subset U$ whenever $A \subset U$ and $U$ is $\mu$-open. It is known that every $\mu$-closed set in a GTS $(X, \mu)$ is $\mu g$-closed, but reverse implication is not true in general. The complement of a $\mu g$-closed set is called $\mu g$-open. The union of all $\mu g$-open subsets of a subset $A$ of $(X, \mu)$ is called the $\mu g$-interior of $A$ and denoted by $\text{int}_{\mu g}(A)$. The intersection of all $\mu g$-closed sets containing a subset $A$ is called the $\mu g$-closure of $A$ and denoted by $\text{cl}_{\mu g}(A)$. If $A$ is $\mu g$-closed, then $A = \text{cl}_{\mu g}(A)$. The converse does not hold in general.

The family of all $\mu g$-open (respectively $\mu g$-closed, $\mu$-closed) sets of $(X, \mu)$ is denoted by $GO(\mu)$ (respectively $GC(\mu)$, $C(\mu)$). The family of all $\mu g$-open (respectively $\mu g$-closed, $\mu$-closed) sets containing a point $x \in X$ is denoted by $GO(\mu, x)$ (respectively $GC(\mu, x)$, $C(\mu, x)$).

**Definition 3.** A function $f : (X, \mu) \to (Y, \lambda)$, where $(X, \mu)$ and $(Y, \lambda)$ are two GTS’s, is called:

(a) $(\mu g, \lambda)$-continuous [12] if $f^{-1}(V)$ is $\mu g$-closed in $(X, \mu)$ for each $\lambda$-closed set $V$ in $(Y, \lambda)$,

(b) $(\mu g, \lambda g)$- irresolute [12] if $(\mu, \lambda)$- irresolute function) if $f^{-1}(V)$ is $\mu g$-closed $(\mu$-closed) in $(X, \mu)$ for each $\lambda g$-closed $(\lambda$-closed set) $V$ in $(Y, \lambda)$,

(c) contra $(\mu, \lambda)$-continuous [7] if $f^{-1}(V)$ is $\mu$-closed in $(X, \mu)$ for each $\lambda$-open set $V$ in $(Y, \lambda)$,

(d) contra $(\mu g, \lambda)$-continuous $f^{-1}(V)$ is $\mu g$-closed in $(X, \mu)$ for each $\lambda$-open set $V$ in $(Y, \lambda)$,

(e) $(\mu, \lambda)$-closed if $f(F)$ is $\lambda$-closed in $(Y, \lambda)$ for each $\mu$-closed set $F$ in $(X, \mu)$.

**Remark 1.** Assume that $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous. Since $\emptyset \in \lambda$ and $f$ is contra $(\mu g, \lambda)$-continuous, $f^{-1}(\emptyset) = \emptyset$ is $\mu g$-closed and this implies that $\emptyset$ is $\mu$-closed, because it is true that $\emptyset \subset \emptyset \in \mu$ and $\text{cl}_{\mu}(\emptyset) \subset \emptyset \subset \text{cl}_{\mu}(\emptyset)$. So, if $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu g, \lambda)$-continuous, then $(X, \mu)$ is a strong GTS. Same is true for if $f : (X, \mu) \to (Y, \lambda)$ is contra $(\mu, \lambda)$-continuous.

The concept of contra $(\mu g, \lambda)$-continuous functions is a generalization of Contra $sg$-Continuous Maps [8].

**Definition 4.** A GTS $(X, \mu)$ is called:

(a) $\mu$-Urysohn if for each pair of distinct points $x$ and $y$ in $X$, there exist $\mu$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $c_{\mu}(U) \cap c_{\mu}(V) = \emptyset$. 

(b) $\mu$-$T_{1\frac{1}{2}}$-space [9] if each $\mu g$-closed subset of $(X, \mu)$ is $\mu$-closed,

(c) $\mu g$-connected [12] if $X$ cannot be written as a disjoint union of two nonempty $\mu g$-open sets,

(d) weakly $\mu$-Hausdorff (see [13]) if each element of $X$ is an intersection of $\mu r$-closed sets.

**Definition 5** ([9]). Let $A$ be a subset of a GTS $(X, \mu)$. The set $\cap \{U \in \mu : A \subseteq U\}$ is called the $\mu$-kernel of $A$ and denoted by $\mu$-ker($A$).

The following Lemma due to D. Jayanthi stated without proof in [5], we give the proofs for the sake of completeness.

**Lemma 1** ([5]). Let $(X, \mu)$ be a GTS and $A, B \subseteq X$. The following properties hold:

(a) $\mu$-ker($A$) $\supset$ $A$ and if $A \in \mu$ then $A = \mu$-ker($A$)

(b) If $A \subseteq B$, then $\mu$-ker($A$) $\subseteq$ $\mu$-ker($B$).

(c) $x \in \mu$-ker($A$) iff $A \cap F \neq \emptyset$ for any $\mu$-closed set $F$ containing $x$.

**Proof.** (a) Let $U_A = \{O : A \subseteq O \in \mu\}$ be the family of all $\mu$-open sets containing $A$. Then we have $\mu$-ker($A$) $=$ $\bigcap U_A \supset A$. If $A \in \mu$, then $A \in U_A$ and $\bigcap U_A \subseteq A$ gives $A = \bigcap U_A = \mu$-ker($A$).

(b) Let $A \subseteq B$, consider $U_A = \{U : A \subseteq U \in \mu\}$ and $U_B = \{U : B \subseteq U \in \mu\}$. Then for $U \in U_B$, it is true that $A \subseteq B \subseteq U \in U_B$, that is $U \in U_A$ and $U_B \subseteq U_A$ and this implies $[(U_A - U_B) \cup U_B] = U_A$, so we have

$$\mu\text{-}ker(A) = \bigcap U_A$$

$$= (\bigcap (U_A - U_B)) \cap (\bigcap U_B)$$

$$\subseteq \bigcap U_B$$

$$= \mu\text{-}ker(B).$$

(c) Let $x \in \mu$-ker($A$) and suppose that $A \cap F = \emptyset$ for some $\mu$-closed set $F$ containing $x$. Then $A \subseteq X - F \in \mu$, and $x \notin X - F$ but we have

$$x \in \mu\text{-}ker(A) \subseteq \mu\text{-}ker(X - F) = X - F$$

which is a contradiction.

Conversely, assume that $A \cap F \neq \emptyset$ for any $\mu$-closed set $F$ containing $x$, but $x \notin \mu$-ker($A$). Then, there exists a $\mu$-open set $V$ such that $A \subseteq V$ and $x \notin V$. Thus we have $x \in X - V \subset (X - A)$ and $X - V$ is $\mu$-closed. But this implies $(X - V) \cap A \subset (X - A) \cap A = \emptyset$ that is $(X - V) \cap A = \emptyset$, but this contradicts with the hypothesis.
3. Characterizations of contra \((\mu g, \lambda)\)-continuous functions

**Remark 2.** From the definitions we have stated above, we observe that in a \(GTS (X, \mu)\), every contra \((\mu, \lambda)\)-continuous function is contra \((\mu g, \lambda)\)-continuous. However the converse does not hold in general.

**Example 1.** Let \(\mathbb{R}\) be the set of real numbers, \(\mu = \{\mathbb{R}, \emptyset, \mathbb{R}\setminus\{0\}, \mathbb{R}\setminus\{-1, 1\}\}\) and \(\lambda = \{\emptyset, \{1\}, \mathbb{R}\}\). Let \(f : (\mathbb{R}, \mu) \rightarrow (\mathbb{R}, \lambda)\) be the identity function. Then \(f\) is contra \((\mu g, \lambda)\)-continuous but not contra \((\mu, \lambda)\)-continuous.

**Proposition 1.** Let \(f : (X, \mu) \rightarrow (Y, \lambda)\) be a function. Suppose that \((X, \mu)\) is \(\mu\)-\(T_1\)-space. Then the following properties are equivalent:

(i) \(f\) is contra \((\mu g, \lambda)\)-continuous,

(ii) \(f\) is contra \((\mu, \lambda)\)-continuous.

**Proof.** This is clear. 

**Theorem 1.** Suppose that \(GC(\mu)\) is closed under arbitrary intersections. Then the following are equivalent for a function \(f : (X, \mu) \rightarrow (Y, \lambda) :\)

(a) \(f\) is contra \((\mu g, \lambda)\)-continuous,

(b) The inverse image of each \(\lambda\)-closed set in \((Y, \lambda)\) is \(\mu g\)-open.

(c) For each \(x \in X\) and each \(\lambda\)-closed set \(B\) containing \(f(x)\), there exists a \(\mu g\)-open set \(A\) in \(X\) such that \(x \in A\) and \(f(A) \subset B\),

(d) \(f(cl_{\mu g}(A)) \subset \lambda\)-ker\((f(A))\) for every subset \(A\) of \(X\),

(e) \(cl_{\mu g}(f^{-1}(B)) \subset f^{-1}(\lambda\)-ker\((B))\) for every subset \(B\) of \(Y\).

**Proof.** (a) \(\Rightarrow\) (b): Let \(G\) be a \(\lambda\)-closed set in \(Y\). Then \(Y \setminus G\) is \(\lambda\)-open and by (a), \(f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)\) is \(\mu g\)-closed. Thus \(f^{-1}(G)\) is \(\mu g\)-open.

(b) \(\Rightarrow\) (a): Let \(U \in \lambda\). Then \(Y \setminus U\) is \(\lambda\)-closed and by (b), \(f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)\) is \(\mu g\)-open, thus \(f^{-1}(U)\) is \(\mu g\)-closed. Hence, \(f\) is contra \((\mu g, \lambda)\)-continuous.

(c) \(\Rightarrow\) (b): Let \(B\) be a \(\lambda\)-closed set with \(x \in f^{-1}(B)\). Since \(f(x) \in B\), by (c) there exists a \(\mu g\)-open set \(A\) containing \(x\) such that \(f(A) \subset B\). It follows that \(x \in A \subset f^{-1}(B)\). Hence, \(f^{-1}(B)\) is \(\mu g\)-open.

(b) \(\Rightarrow\) (d): Let \(A\) be any subset of \(X\) and \(y \notin \lambda\)-ker\((f(A))\). Then by Lemma 1, there exists a \(\lambda\)-closed set \(F\) containing \(y\) such that \(f(A) \cap F = \emptyset\). Hence, we have \(A \cap f^{-1}(F) = \emptyset\) and \(cl_{\mu g}(A) \cap f^{-1}(F) = \emptyset\). Thus we obtain, \(f(cl_{\mu g}(A)) \cap F = \emptyset\) and \(y \notin f(cl_{\mu g}(A))\). Therefore, \(f(cl_{\mu g}(A)) \subset \lambda\)-ker\((f(A))\).
(d) \implies (e): Let B be any subset of Y. By (d) we have
\[ f(cl_{\mu}(f^{-1}(B))) \subseteq \lambda \ker(f(f^{-1}(B))) \subseteq \lambda \ker(B). \]
and this implies
\[ cl_{\mu}(f^{-1}(B)) \subseteq f^{-1}(f(cl_{\mu}(f^{-1}(B)))) \subseteq f^{-1}(\lambda \ker(B)) \]
Then we have the result \( cl_{\mu}(f^{-1}(B)) \subseteq f^{-1}(\lambda \ker(B)). \)

(e) \implies (a): Let B \in \lambda, then by (e), \( cl_{\mu}(f^{-1}(B)) \subseteq f^{-1}(\lambda \ker(B)) = f^{-1}(B) \) and \( cl_{\mu}(f^{-1}(B)) = f^{-1}(B) \). Since GC(\mu) is closed under arbitrary intersections, \( f^{-1}(B) \) is \( \mu \)-closed in \((X, \mu)\).

**Notation 1.** Let \((X, \mu)\) and \((Y, \kappa)\) be generalized topological spaces, and let \( U = \{ U \times V : U \in \mu, V \in \kappa \} \). It is known that \( U \) generates a generalized topology \( \nu = \mu \times \kappa \) on \( X \times Y \), called the generalized product topology ([4], [11]) on \( X \times Y \), that is, \( \nu = \{ \) all possible unions of members of \( U \}\)

**Theorem 2.** Let \( f : (X, \mu) \to (Y, \lambda) \) be a function and \( g : (X, \mu) \to (X \times Y, \nu) \) be the graph function of \( f \), defined by \( g(x) = (x, f(x)) \) for every \( x \in X \). If \( g \) is contra \((\mu, \nu)\)-continuous, then \( f \) is \((\mu, \lambda)\)-continuous.

**Proof.** Let \( U \) be any \( \lambda \)-open set in \((Y, \lambda)\). By remark 1, \((X, \mu)\) is strong GTS, hence \( X \times U \) is a \( \nu \)-open set in \( X \times Y \). It follows that \( f^{-1}(U) = g^{-1}(X \times U) \) is \( \mu \)-closed. Thus, \( f \) is contra \((\mu, \lambda)\)-continuous.

**Definition 6.** For a function \( f : (X, \mu) \to (Y, \lambda), \) the subset \( \{(x, f(x)) : x \in X\} \subseteq X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \).

**Definition 7.** Let \((X, \mu)\) and \((Y, \lambda)\) are two GTS’s, consider \( \nu \) as generalized product space of the \( \mu \) and \( \lambda \) on \( X \times Y \). The graph \( G(f) \) of a function \( f : (X, \mu) \to (Y, \lambda) \) is said to be contra \( \nu \mu \)-closed graph if for each \((x, y) \in (X \times Y) \setminus G(f), \) there exists a \( \mu \)-open set \( U \) in \( X \) containing \( x \) and a \( \lambda \)-closed set \( V \) in \( Y \) containing \( y \) such that \((U \times V) \cap G(f) = \emptyset \).

**Proposition 2.** The following properties are equivalent for the graph \( G(f) \) of a function \( f : (X, \mu) \to (Y, \lambda) : \)

(a) \( G(f) \) is contra \( \nu \mu \)-closed graph,
(b) For each \((x, y) \in (X \times Y) \setminus G(f), \) there exists a \( \mu \)-open set \( U \) in \( X \) containing \( x \) and a \( \lambda \)-closed set \( V \) in \( Y \) containing \( y \) such that \( f(U) \cap V = \emptyset \).

**Proof.** (a) \( \implies \) (b): Let \((x, y) \in (X \times Y) \setminus G(f). \) By (a), there exists a \( \mu \)-open set \( U \) in \( X \) containing \( x \) and a \( \lambda \)-closed set in \( Y \) containing \( y \)
such that \((U \times V) \cap G(f) = \emptyset\). Since \((x, y) \notin G(f), x \in U, y \in V\) we have \(f(x) \neq y\) and therefore \(f(U) \cap V = \emptyset\).

\((b) \implies (a)\): Let \((x, y) \in (X \times Y) \setminus G(f)\). By \((b)\), there exists a \(\mu\)-open set \(U\) in \(X\) containing \(x\) and a \(\lambda\)-closed set \(V\) in \(Y\) containing \(y\) such that \(f(U) \cap V = \emptyset\). Hence, \((x, y) \in (U \times V) \subset (X \times Y) \setminus G(f)\). ■

**Theorem 3.** If \(f : (X, \mu) \to (Y, \lambda)\) is contra \((\mu, \lambda)\)-continuous function and \((Y, \lambda)\) is \(\lambda\)-Urysohn, then \(G(f)\) is contra \(\nu\)-closed.

**Proof.** Let \((x, y) \in (X \times Y) \setminus G(f)\). It follows that \(f(x) \neq y\). Since \((Y, \lambda)\) is \(\lambda\)-Urysohn, there exist \(\lambda\)-open sets \(B\) and \(C\) such that \(f(x) \in B, y \in C\) and \(c_\lambda(B) \cap c_\lambda(C) = \emptyset\). Since \(f\) is contra \((\mu, \lambda)\)-continuous, there exists a \(\mu\)-open set \(A\) in \(X\) containing \(x\) such that \(f(A) \subset c_\lambda(B)\). Therefore, \(f(A) \cap c_\lambda(C) = \emptyset\) and \(G(f)\) is contra \(\nu\)-closed graph in \(X \times Y\). ■

**Theorem 4.** Let \(\{(X_i, \mu_i) : i \in I\}\) be any family of strong GTS’ s. If \(f : (X, \mu) \to (\Pi X_i, \nu)\) is contra \((\mu, \nu)\)-continuous, then \(p_i \circ f : (X, \mu) \to (X_i, \mu_i)\) is contra \((\mu, \mu_i)\)-continuous for each \(i \in I\), where \(p_i\) is the projection of \((\Pi X_i, \nu)\) onto \((X_i, \mu_i)\).

**Proof.** We shall consider a fixed \(i \in I\). Suppose \(U_i\) is an arbitrary \(\mu_i\)-open set of \(X_i\). Since each \((X_i, \mu_i)\) is strong GTS, \(p_i\) is \((\nu, \mu_i)\)-continuous by Proposition 2.7 of [4], that is \(p_i^{-1}(U_i)\) is \(\nu\)-open in \((\Pi X_i, \nu)\). Since \(f\) is contra \((\mu, \nu)\)-continuous, we have \(f^{-1}(p_i^{-1}(U_i)) = (p_i \circ f)^{-1}(U_i)\) is \(\mu_i\)-\(\nu\)-closed. Therefore, \(p_i \circ f\) is contra \((\mu, \mu_i)\)-continuous. ■

**Definition 8.** A GTS \((X, \mu)\) is said to be locally \(\mu\)-indiscrete if every \(\mu\)-open set of \((X, \mu)\) is \(\mu\)-closed.

**Theorem 5.** If \(f : (X, \mu) \to (Y, \lambda)\) is contra \((\mu, \lambda)\)-continuous with \((X, \mu)\) is locally \(\mu\)-indiscrete, then \(f\) is contra \((\mu, \lambda)\)-continuous.

**Proof.** This is clear. ■

**Theorem 6.** Suppose that \((X, \mu)\), \((Y, \lambda)\) are two GTS’s and \(GO(\mu)\) is closed under arbitrary unions. If a function \(f : (X, \mu) \to (Y, \lambda)\) is contra \((\mu, \lambda)\)-continuous and \((Y, \lambda)\) is \(\lambda\)-regular, then \(f\) is \((\mu, \lambda)\)-continuous.

**Proof.** Let \(x\) be an arbitrary point of \((X, \mu)\) and \(V\) be a \(\lambda\)-open set of \(Y\) containing \(f(x)\). Since \((Y, \lambda)\) is \(\lambda\)-regular, there exists a \(\lambda\)-open set \(G\) in \(Y\) containing \(f(x)\) such that \(c_\lambda(G) \subset V\). Because \(f\) is contra \((\mu, \lambda)\)-continuous, there exists \(U \in GO(\mu)\) containing \(x\) such that \(f(U) \subset c_\lambda(G)\). Then \(f(U) \subset c_\lambda(G) \subset V\). Hence, \(f\) is \((\mu, \lambda)\)-continuous. ■
**Theorem 7.** Let \((X, \mu)\) be a \(\mu\)-connected GTS and \((Y, \lambda)\) be any GTS. If there is surjective, contra \((\mu, \lambda)\)-continuous function \(f : (X, \mu) \rightarrow (Y, \lambda)\), then \((Y, \lambda)\) is \(\lambda\)-connected.

**Proof.** Let \(f : (X, \mu) \rightarrow (Y, \lambda)\) be a contra \((\mu, \lambda)\)-continuous, surjective function of a \(\mu\)-connected space \((X, \mu)\) to a GTS \((Y, \lambda)\). Suppose that \((Y, \lambda)\) is \(\lambda\)-disconnected. Let \(A\) and \(B\) form a disconnection of \((Y, \lambda)\). Then \(A\) and \(B\) are \(\lambda\)-open and \(Y = A \cup B\) where \(A \cap B = \emptyset\). Since \(f\) is contra \((\mu, \lambda)\)-continuous and surjective, \(X = f^{-1}(A) \cup f^{-1}(B)\) where \(f^{-1}(A)\) and \(f^{-1}(B)\) are nonempty \(\mu\)-closed sets in \((X, \mu)\). Also \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\), so \(f^{-1}(A)\) and \(f^{-1}(B)\) are \(\mu\)-open. This contradicts with the fact that \((X, \mu)\) is \(\mu\)-connected. Hence \((Y, \lambda)\) is \(\lambda\)-connected. \(\blacksquare\)

**Theorem 8.** Let \((X, \mu)\) be \(\mu\)-connected. Then each contra \((\mu, \lambda)\)-continuous function of \(X\) into a \(\lambda\)-discrete GTS \((Y, \lambda)\) with at least two points is a constant function.

**Proof.** Let \(f : (X, \mu) \rightarrow (Y, \lambda)\) be a contra \((\mu, \lambda)\)-continuous function and \((X, \mu)\) be a \(\mu\)-connected GTS. Then \((X, \mu)\) is covered by \(\mu\)-open and \(\mu\)-closed covering \(\{f^{-1}\{\{y\}\} : y \in Y\}\). By assumption, \(f^{-1}\{\{y\}\} = \emptyset\) or \(X\) for each \(y \in Y\). If \(f^{-1}\{\{y\}\} = \emptyset\) for all \(y \in Y\), then it fails to be a function. Then there exists only one point \(y \in Y\) such that \(f^{-1}\{\{y\}\} \neq \emptyset\) and hence \(f^{-1}\{\{y\}\} = X\) which shows that \(f\) is a constant function. \(\blacksquare\)

**Theorem 9.** If \(f\) is a contra \((\mu, \lambda)\)-continuous function from a \(\mu\)-connected GTS \((X, \mu)\) onto a GTS \((Y, \lambda)\), then \(Y\) is not a \(\lambda\)-discrete space.

**Proof.** Suppose that \((Y, \lambda)\) is \(\lambda\)-discrete. Let \(A\) be a proper nonempty \(\lambda\)-open and \(\lambda\)-closed subset of \((Y, \lambda)\). Then \(f^{-1}(A)\) is a proper nonempty \(\mu\)-open subset of \((X, \mu)\), which is a contradiction with the fact that \((X, \mu)\) is \(\mu\)-connected. \(\blacksquare\)

**Definition 9.** A GTS \((X, \mu)\) is said to be \(\mu\)-normal (resp. \(\mu\)-normal [10]) if each pair of nonempty \(\mu\)-closed sets can be separated by disjoint \(\mu\)-open (resp. \(\mu\)-open ) sets.

**Theorem 10.** If \(f : (X, \mu) \rightarrow (Y, \lambda)\) is a contra \((\mu, \lambda)\)-continuous, \((\mu, \lambda)\)-closed, injection and \((Y, \lambda)\) is \(\lambda\)-normal, then \((X, \mu)\) is \(\mu\)-normal.

**Proof.** Let \(F_1, F_2\) be disjoint \(\mu\)-closed subsets of \((X, \mu)\). Since \(f\) is \((\mu, \lambda)\)-closed and injective, \(f(F_1)\) and \(f(F_2)\) are disjoint \(\lambda\)-closed subset of \((Y, \lambda)\). \(f(F_1)\) and \(f(F_2)\) are separated by disjoint \(\lambda\)-open sets \(V_1, V_2\), respectively, because \((Y, \lambda)\) is \(\lambda\)-normal. Hence, \(F_i \subset f^{-1}(V_i)\) and \(f^{-1}(V_i)\) is \(\mu\)-open in \((X, \mu)\) for \(i = 1, 2\) and \(f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset\). Thus, \((X, \mu)\) is \(\mu\)-normal. \(\blacksquare\)
4. Composition properties

**Remark 3.** Let \( f : (X, \mu) \to (Y, \lambda) \) be contra \((\mu g, \lambda)\)-continuous and \( g : (Y, \lambda) \to (Z, \nu) \) be contra \((\lambda g, \nu)\)-continuous. Then, the composition \( g \circ f : (X, \mu) \to (Z, \nu) \) need not be contra \((\mu g, \nu)\)-continuous.

**Example 2.** Let \( \mathbb{R} \) be the set of real numbers, \( \mu = \{\emptyset, \mathbb{R}\setminus\{-1\}, \mathbb{R}\setminus\{1\}, \mathbb{R}\setminus\{-1, 1\}, \mathbb{R}\} \), \( \lambda = \{\emptyset, \mathbb{R}\setminus\{0\}, \mathbb{R}\setminus\{0, 1\}, \mathbb{R}\} \) and \( \nu = \{\emptyset, \mathbb{R}\setminus\{-1, 1\}, \mathbb{R}\} \). Then the identity function \( f : (\mathbb{R}, \mu) \to (\mathbb{R}, \lambda) \) is contra \((\mu g, \lambda)\)-continuous and the identity function \( g : (\mathbb{R}, \lambda) \to (\mathbb{R}, \nu) \) is contra \((\lambda g, \nu)\)-continuous. But the composition \( g \circ f : (\mathbb{R}, \mu) \to (\mathbb{R}, \nu) \) is not contra \((\mu g, \nu)\)-continuous.

**Theorem 11.** Let \( (X, \mu), (Z, \nu) \) be two GTS’s and \((Y, \lambda)\) be a \(\lambda\)-\(T^*_2\)-space. Let \( f : (X, \mu) \to (Y, \lambda) \) be \((\mu, \lambda)\)-irresolute function and \( g : (Y, \lambda) \to (Z, \nu) \) be contra \((\lambda g, \nu)\)-continuous. Then \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous.

**Proof.** Let \( F \) be any \( \nu \)-open subset of \((Z, \nu)\). Since \( g \) is contra \((\lambda g, \nu)\)-continuous, \( f^{-1}(F) \) is \( \lambda g \)-closed in \((Y, \lambda)\). But \((Y, \lambda)\) is \(\lambda\)-\(T^*_2\)-space, so \( f^{-1}(F) \) is \( \lambda \)-closed. Since \( f \) is \((\mu, \lambda)\)-irresolute, \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( \mu \)-closed. Since every \( \mu \)-closed set in a GTS \((X, \mu)\) is \(\mu g\)-closed, \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous.

**Theorem 12.** Let \( f : (X, \mu) \to (Y, \lambda) \) be \((\mu g, \lambda g)\)-irresolute function and \( g : (Y, \lambda) \to (Z, \nu) \) be contra \((\lambda g, \nu)\)-continuous function. Then \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous.

**Proof.** Let \( F \) be a \( \nu \)-open set in \((Z, \nu)\). Then \( g^{-1}(F) \) is \( \lambda g \)-closed in \((Y, \lambda)\), because \( g \) is contra \((\lambda g, \nu)\)-continuous. Since \( f \) is \((\mu g, \lambda g)\)-irresolute, \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is \( \mu g \)-closed. Thus, \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous.

**Corollary 1.** Let \( f : (X, \mu) \to (Y, \lambda) \) be \((\mu g, \lambda g)\)-irresolute and \( g : (Y, \lambda) \to (Z, \nu) \) be contra \((\lambda, \nu)\)-continuous function. Then \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous.

**Definition 10.** A function \( f : (X, \mu) \to (Y, \lambda) \) is said to be pre-(\(\mu g, \lambda g\))-open if the image of every \(\mu g\)-open set is \(\lambda g\)-open.

**Theorem 13.** Let \( f : (X, \mu) \to (Y, \lambda) \) be surjective, \((\mu g, \lambda g)\)-irresolute, pre-(\(\mu g, \lambda g\))-open function and \( g : (Y, \lambda) \to (Z, \nu) \) be any function. Then \( g \circ f : (X, \mu) \to (Z, \nu) \) is contra \((\mu g, \nu)\)-continuous if \( g \) is contra \((\lambda g, \nu)\)-continuous.

**Proof.** Let \( g : (Y, \lambda) \to (Z, \nu) \) be a contra \((\lambda g, \nu)\)-continuous function and \( F \) be a \( \nu \)-open subset of \((Z, \nu)\). Since \( g \) is contra \((\lambda g, \nu)\)-continuous,
$g^{-1}(F)$ is $\lambda g$-closed. But $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\mu g$-closed because $f$ is $(\mu g, \lambda g)$-irresolute. Thus, $g \circ f : (X, \mu) \to (Z, \nu)$ is contra $(\mu g, \nu)$-continuous.

Conversely, let $g \circ f : (X, \mu) \to (Z, \nu)$ be contra $(\mu g, \nu)$-continuous and let $F$ be a $\nu$-closed subset of $(Z, \nu)$. Then, $(g \circ f)^{-1}(F)$ is a $\mu g$-open. Since $f$ is pre-$(\mu g, \lambda g)$-open and surjective, $f(f^{-1}(g^{-1}(F))) = g^{-1}(F)$ is $\lambda g$-open. Hence, $g : (Y, \lambda) \to (Z, \nu)$ is contra $(\lambda g, \nu)$-continuous.

**Theorem 14.** If $f : (X, \mu) \to (Y, \lambda)$ is $(\mu g, \lambda g)$-irresolute function with $(Y, \lambda)$ as locally $\lambda g$-indiscrete space and $g : (Y, \lambda) \to (Z, \nu)$ is contra $(\lambda g, \nu)$-continuous function, then $g \circ f : (X, \mu) \to (Z, \nu)$ is $(\mu g, \nu)$-continuous.

**Proof.** Let $F$ be a $\nu$-closed subset of $(Z, \nu)$. Since, $g$ is contra $(\lambda g, \nu)$-continuous, $g^{-1}(F)$ is $\lambda g$-open in $(Y, \lambda)$. But $(Y, \lambda)$ is locally $\lambda g$-indiscrete, so $g^{-1}(F)$ is $\lambda g$-closed. Since $f$ is $(\mu g, \lambda g)$-irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\mu g$-closed. Therefore, $g \circ f$ is $(\mu g, \nu)$-continuous.

5. Some covering and separation properties

**Definition 11.** A GTS $(X, \mu)$ is said to be

(a) $\mu g$-compact if every $\mu g$-open cover of $(X, \mu)$ has a finite subcover,

(b) strongly $\mu$-$S$-closed if every $\mu$-closed cover of $(X, \mu)$ has a finite subcover,

(c) countably $\mu g$-compact if every countable cover of $(X, \mu)$ by $\mu g$-open sets has a finite subcover,

(d) strongly countably $\mu$-$S$-closed if every countable cover of $(X, \mu)$ by $\mu$-closed sets has a finite subcover,

(e) $\mu g$-Lindelöf if every $\mu g$-open cover of $(X, \mu)$ has a countable subcover,

(f) strongly $\mu$-$S$-Lindelöf if every $\mu$-closed cover of $(X, \mu)$ has a countable subcover.

**Theorem 15.** The surjective contra $(\mu g, \lambda)$-continuous image of a $\mu g$-compact (resp. $\mu g$-Lindelöf, countably $\mu g$-compact) space is strongly $\lambda$-$S$-closed (resp. strongly $\lambda$-$S$-Lindelöf, strongly countable $\lambda$-$S$-closed).

**Proof.** Suppose that $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous surjection. Let $\{V_\alpha : \alpha \in \nabla\}$ be any $\lambda$-closed cover of $(Y, \lambda)$. Since $f$ is contra $(\mu g, \lambda)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a $\mu g$-open cover of $X$ and hence there exists a finite subset $\nabla_0$ of $\nabla$ such that $X = \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$. Therefore we have, $Y = \bigcup_{\alpha \in \nabla_0} V_\alpha$ and $(Y, \lambda)$ is strongly $\lambda$-$S$-closed.

The other proofs can be obtained similarly.

**Definition 12.** A GTS $(X, \mu)$ is said to be
(a) $\mu g$-closed-compact if every $\mu g$-closed cover of $(X, \mu)$ has a finite subcover,
(b) countably $\mu g$-closed compact if every countable cover of $(X, \mu)$ by $\mu g$-closed sets has a finite subcover,
(c) $\mu g$-closed-Lindelöf if every $\mu g$-closed cover of $(X, \mu)$ has a countable subcover.

Theorem 16. Surjective, contra $(\mu g, \lambda)$-continuous image of a $\mu g$-closed compact (resp. $\mu g$-closed Lindelöf, countably $\mu g$-closed compact) space is $\lambda$-compact (resp. $\lambda$-Lindelöf, countably $\lambda$-compact).

Proof. Suppose that $f : (X, \mu) \to (Y, \lambda)$ is a contra $(\mu g, \lambda)$-continuous surjection. Let $\{V_\alpha : \alpha \in \nabla\}$ be any $\lambda$-open cover of $(Y, \lambda)$. Since $f$ is contra $(\mu g, \lambda)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a $\mu g$-closed cover of $(X, \mu)$, hence there exists a finite subset $\nabla_0$ of $\nabla$ such that $X = \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha)$. Therefore we have $Y = \bigcup_{\alpha \in \nabla_0} V_\alpha$ and $Y$ is $\lambda$-compact. The other proofs can be obtained similarly. $\blacksquare$

Definition 13. A GTS $(X, \mu)$ is said to be $\mu g$-$T_1$ if for each pair of distinct points $x$ and $y$ in $(X, \mu)$, there exist $\mu g$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $y \notin U$ and $x \notin V$.

Definition 14. A GTS $(X, \mu)$ is said to be $\mu g$-$T_2$ if for each pair of distinct points $x$ and $y$ in $(X, \mu)$, there exist disjoint $\mu g$-open sets $U$ and $V$ containing $x$ and $y$ respectively.

Theorem 17. Let $(X, \mu), (Y, \lambda)$ be two GTS’s. If

(a) for each pair of distinct points $x$ and $y$ in $(X, \mu)$, there exists a function $f$ of $X$ on to $Y$ such that $f(x) \neq f(y)$,
(b) $(Y, \lambda)$ is $\lambda$-Urysohn space, and
(c) $f$ is contra $(\mu g, \lambda)$-continuous at $x$ and $y$.

Then $(X, \mu)$ is $\mu g$-$T_2$.

Proof. Let $x$ and $y$ be distinct points in $(X, \mu)$, from the hypothesis by (b) there exists a $\lambda$-Urysohn space $(Y, \lambda)$, by (a) there exists a function $f : (X, \mu) \to (Y, \lambda)$ such that $f(x) \neq f(y)$ and by (c) $f$ is contra $(\mu g, \lambda)$-continuous at $x$ and $y$. Let $v = f(x)$ and $w = f(y)$, then $v \neq w$. Since $(Y, \lambda)$ is $\lambda$-Urysohn, there exists $\lambda$-open sets $V$ and $W$ containing $v$ and $w$ respectively, such that $c_\lambda(V) \cap c_\lambda(W) = \emptyset$. Since $f$ is contra $(\mu g, \lambda)$-continuous at $x$ and $y$, there exist $\mu g$-open sets $A$ and $B$ containing $x$ and $y$ respectively, such that $f(A) \subset c_\lambda(V)$ and $f(B) \subset c_\lambda(W)$. We have $A \cap B = \emptyset$ since $c_\lambda(V) \cap c_\lambda(W) = \emptyset$. Hence, $(X, \mu)$ is $\mu g$-$T_2$. $\blacksquare$
**Theorem 18.** If \( f : (X, \mu) \rightarrow (Y, \lambda) \) is a contra \((\mu, \lambda)\)-continuous injection and \((Y, \lambda)\) is weakly \(\lambda\)-Hausdorff, then \((X, \mu)\) is \(\mu\)-T\(_1\).

**Proof.** Suppose that \((Y, \lambda)\) is weakly \(\lambda\)-Hausdorff, then for any pair of distinct points \(x \neq y\) in \((X, \mu)\), there exist \(\lambda\)-closed sets \(A, B\) in \((Y, \lambda)\) such that \(f(x) \in A\), \(f(x) \notin B\) and \(f(y) \in B\), \(f(y) \notin A\). Since \(f\) is contra \((\mu, \lambda)\)-continuous, \(f^{-1}(A)\) and \(f^{-1}(B)\) are \(\mu\)-open subsets of \((X, \mu)\) such that \(x \in f^{-1}(A)\), \(x \notin f^{-1}(B)\) and \(y \in f^{-1}(B)\), \(y \notin f^{-1}(A)\). Hence, \((X, \mu)\) is \(\mu\)-T\(_1\).

**Theorem 19.** Let \( f : (X, \mu) \rightarrow (Y, \lambda) \) have a contra \((\mu, \lambda)\)-closed graph. If \( f \) is injective, then \((X, \mu)\) is \(\mu\)-T\(_1\).

**Proof.** Let \(x\) and \(y\) be distinct points in \((X, \mu)\). Then we have \((x, f(y)) \in (X \times Y) \setminus G(f)\). Then, there exists a \(\mu\)-open set \(U\) in \((X, \mu)\) containing \(x\) and a \(\lambda\)-closed set \(F\) containing \(f(y)\) such that \(f(U) \cap F = \emptyset\). Hence, \(U \cap f^{-1}(F) = \emptyset\). Therefore, we have \(y \notin U\). This implies \((X, \mu)\) is \(\mu\)-T\(_1\).

**Theorem 20.** Let \( f : (X, \mu) \rightarrow (Y, \lambda) \) be a contra \((\mu, \lambda)\)-continuous injection. If \((Y, \lambda)\) is ultra \(\lambda\)-Hausdorff, then \((X, \mu)\) is \(\mu\)-T\(_2\).

**Proof.** Let \(x\) and \(y\) be two distinct points in \((X, \mu)\). Then \(f(x) \neq f(y)\) and there exist \(\lambda\)-clopen sets \(A, B\) containing \(f(x)\), \(f(y)\) respectively, such that \(A \cap B = \emptyset\). Since \(f\) is contra \((\mu, \lambda)\)-continuous, then \(f^{-1}(A)\), \(f^{-1}(B)\) are \(\mu\)-open sets such that \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\). Hence, \((X, \mu)\) is \(\mu\)-T\(_2\).

**References**


Uğur Şengül
Department of Mathematics
Faculty of Science and Letters
Marmara University
34722, Göztepe-İstanbul, Turkey
e-mail: usengul@marmara.edu.tr

Seda Nur Dündar
Institute of Pure and Applied Sciences
Marmara University
34722, Göztepe-İstanbul, Turkey
e-mail: seda.n. Dundar@gmail.com

Received on 17.02.2017 and, in revised form, on 03.11.2017.