ON SOME RECENT FIXED POINT RESULTS FOR SINGLE AND MULTI-VALUED MAPPINGS IN $b$-METRIC SPACES

Abstract. The main purpose of this paper is to improve and correct some results in $b$-metric spaces. Moreover, we prove that some results can be slightly relaxed and also we explore some proof techniques which provide short proofs of the results.

Key words: $b$-metric space, $b$-complete, $b$-Cauchy, $b$-continuous, Picard sequence, multi-valued mapping.

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1. Definitions, notations and preliminaries

We start our exposition with the next result which will prove extremely useful in the sequel.

Lemma 1 ([15]). Let $(X, d, s)$ be a $b$-metric space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in $X$. If there exists $\gamma \in [0, 1)$ such that

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$, then $\{x_n\}_{n \in \mathbb{N}}$ is a $b$-Cauchy sequence.

Otherwise, for more details on $b$-metric spaces we refer the reader to ([1]–[5], [8]–[11], [13]–[15], [17]–[21]).

In [6], authors proved some fixed point theorems in $b$-metric spaces. We will restrict our attention to the following two results.

Theorem 1 ([6] Theorem 1). Let $(X, d, s \geq 1)$ be a complete $b$-metric space and define the sequence $\{x_n\}$ in $X$ by the recursion

$$x_n = T x_{n-1} = T^n x_0.$$
Let \( T : X \to X \) be a mapping such that

\[
d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 [d(y, Tx) + d(x, Ty)]
\]

for all \( x, y \in X \), where \( \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1 \).

Then there exists \( x^* \in X \) such that \( x_n \to x^* \) and \( x^* \) is a unique fixed point.

**Remark 1.** The condition \( \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1 \) should be replaced by

\[
\lambda_i \geq 0, \; i = 1, 4, \quad \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1.
\]

Indeed, for \( \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_1 = 1 \) and \( s = 1 \), we have

\[
d(Tx, Ty) \leq d(x, y), \quad x, y \in X.
\]

For \( T = I_X \), Theorem 1 is not valid, since the fixed point of \( T \) is not unique.

If \( s = 1 \), then \((X, d)\) is a metric space and the condition (1) implies

\[
d(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2} \right\},
\]

where \( k = \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1 \). Note that

\[
\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 \leq \lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4 < 1.
\]

With (2) we recover the well known result for generalized Ćirić contraction map in the metric space and obtain a unique fixed point.

**Remark 2.** Let us note that the condition \( \lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 < 1 \) from [6] implies the condition \( \lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1 \). Now, according to the new condition, we can improve the proof of Theorem 1 from [6]. Firstly, the proof that \( \{x_n\} \) is a \( b \)-Cauchy sequence can be shorter than that in [6]. Indeed, if \( x_n \neq x_{n-1} \), for all \( n \in \mathbb{N} \), we have

\[
d(Tx_{n-1}, Tx_n) \leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_{n-1}, x_n) + \lambda_3 d(x_n, x_{n+1}) + \lambda_4 sd(x_{n-1}, x_n) + \lambda_4 sd(x_n, x_{n+1}),
\]

and also

\[
d(Tx_n, Tx_{n-1}) \leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_{n-1}, x_n) + \lambda_4 sd(x_{n-1}, x_n) + \lambda_4 sd(x_n, x_{n+1}).
\]
It follows easily that $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ where

$$k = \max \left\{ \frac{\lambda_1 + \lambda_3 + s\lambda_4}{1 - \lambda_2 - s\lambda_4}, \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4} \right\} < 1,$$

and according to Lemma 1, we have that $\{x_n\}$ is a $b$-Cauchy sequence.

**Remark 3.** It is not hard to check that the proof of Theorem 1 in [6] is not correct (see pages 3 and 4). Really, the fact that $x^* = \lim_{n \to \infty} x_n$ is the fixed point of $T$ is not clear enough since it was not shown that $k < 1$.

We prove this with the new condition $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$.

$$\frac{1}{s}d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$$

$$\leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, Tx^*) + \lambda_4 d(x^*, x_{n+1}) + \lambda_4 s d(x_n, x^*) + \lambda_4 s d(x^*, Tx^*)$$

$$\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + (\lambda_3 + \lambda_4 s) d(x^*, Tx^*),$$

or

$$\left( \frac{1}{s} - \lambda_3 - \lambda_4 s \right) d(x^*, Tx^*) \leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_2 d(x_n, x_{n+1}) + (\lambda_3 + \lambda_4 s) d(x^*, Tx^*),$$

(3)

Similarly,

$$\frac{1}{s}d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$$

$$\leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_3 d(x_n, x_{n+1}) + (\lambda_2 + \lambda_4 s) d(x^*, Tx^*),$$

that is,

(4)

$$\left( \frac{1}{s} - \lambda_2 - \lambda_4 s \right) d(x^*, Tx^*) \leq (1 + \lambda_4) d(x^*, x_{n+1}) + (\lambda_1 + \lambda_4 s) d(x_n, x^*) + \lambda_3 d(x_n, x_{n+1}).$$
Adding (3) and (4) we obtain
\[
\left(\frac{2}{s} - \lambda_2 - \lambda_3 - 2\lambda_4 s\right) d(x^*, Tx^*) \leq 2 \left(1 + \lambda_4\right) d(x^*, x_{n+1}) + 2 \left(\lambda_1 + \lambda_4 s\right) d(x_n, x^*) + (\lambda_2 + \lambda_3) d(x_n, x_{n+1}) \to 0, \text{ as } n \to \infty.
\]

Hence, we can conclude the following.

**Conclusion.** Theorem 1 from [6] holds if the coefficients \(\lambda_i \geq 0, i = 1, 4\), satisfy at least one of the following conditions:

1. \(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1\) for \(s \in [1, 2]\);
2. \(\frac{2}{s} < \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1\) for \(s \in (2, +\infty)\).

Our approach with the new condition \(\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1\) provides the generalization and improves Theorem 3.7 from [10], that is., Theorem 2.19 from [18].

**Remark 4.** Note that condition (2) for \(s > 1\) implies
\[
d(Tx, Ty) \leq k \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}
\]
where \(k = \lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4\).

**Theorem 2** ([6] Theorem 2). Let \((X, d, s \geq 1)\) be a complete b-metric space. Let \(T : X \to X\) be a mapping for which there exist \(\lambda_1, \lambda_2 \in [0, \frac{1}{3})\) such that
\[
d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 [d(x, Tx) + d(y, Ty)],
\]
for all \(x, y \in X\).

Then there exists \(x^* \in X\) such that \(x_n \to x^*\) and \(x^*\) is the unique fixed point.

**Remark 5.** The proof of Theorem 2 is not correct since \(\frac{\lambda_1 + \lambda_2}{1 - \lambda_2} \leq 1\). We give the improved version of this theorem.

If \(s = 1\), then \((X, d)\) is a metric space and the condition \(\lambda_1 + 2\lambda_2 < 1\) is appropriate for metric spaces.

Let \(s > 1\). By the same method as in [6], we have
\[
d(x_n, x_{n+1}) \leq \frac{\lambda_1 + \lambda_2}{1 - \lambda_2} d(x_{n-1}, x_n) = kd(x_{n-1}, x_n).
\]

Since \(\lambda_1, \lambda_2 \in [0, \frac{1}{3})\), it follows that \(\frac{\lambda_1 + \lambda_2}{1 - \lambda_2} = k < 1\), and using Lemma 1, we can conclude that the sequence \(\{x_n\}\) is a b-Cauchy sequence.
The proof falls naturally into three parts.

**Case 1°.** If $T$ is continuous, then $Tx_n \to Tx^*$ as $n \to \infty$, and $x^* = \lim_{n \to \infty} x_n$ is the fixed point of $T$.

**Case 2°.** If $d$ is continuous, then substituting $x = x_n$ and $y = \lim_{n \to \infty} x_n$ in (5), we obtain

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \leq \lambda_1 d(x_n, x^*) + \lambda_2 [d(x_n, x_{n+1}) + d(x^*, Tx^*)].$$

Letting $n \to \infty$, it follows that

$$d(x^*, Tx^*) \leq \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \lambda_2 d(x^*, Tx^*),$$

i.e. $(1-\lambda_2)d(x^*, Tx^*) \leq 0$. Using the fact that $\lambda_2 \in [0, \frac{1}{s})$, we have $Tx^* = x^*$.

**Case 3°.** Neither 1° nor 2° is satisfied. Then

$$\frac{1}{s}d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \leq d(x^*, x_{n+1}) + \lambda_1 d(x_n, x^*) + \lambda_2 [d(x_n, x_{n+1}) + d(x^*, Tx^*)],$$

i.e. $(\frac{1}{s} - \lambda_2)d(x^*, Tx^*) \leq 0$. We conclude that $T$ has a fixed point $x^* = \lim_{n \to \infty} x_n$ if $\lambda_2 < \frac{1}{s}$.

From what has already been proved, we deduce that $T$ has a fixed point if $\lambda_2 < \min\{\frac{1}{s}, \frac{1}{3}\}$.

Now, we will show that our viewpoint sheds some new light on an interesting new result proved in [7].

**Theorem 3 ([7] Theorem 2.2).** Let $(X, d, s \geq 1)$ be a complete b-metric space and $T, S$ self-mappings on $X$ which satisfy

$$d (Sx, Ty) \leq a_1 d (x, Sx) + a_2 d (y, Ty) + a_3 d (x, Ty) + a_4 d (y, Sx) + a_5 d (x, y),$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5$ are nonnegative real numbers satisfying:

(i) $s^2 a_1 + s^2 a_2 + s^3 a_3 + s^3 a_4 + s^2 a_5 < 1$,

(ii) $a_1 = a_2$ or $a_3 = a_4$.

Then $S$ and $T$ have a unique common fixed point.

**Remark 6.** If $s = 1$, then $(X, d)$ is a metric space with the assumptions

$$a_1 + a_2 + a_3 + a_4 + a_5 < 1, \quad a_1 = a_2 \quad \text{or} \quad a_3 = a_4.$$

It follows immediately that the condition (ii) is superfluous.
We will also prove that [[7] Theorem 2.4] is still true if we drop the assumption of function \( \varphi \). We repeat the relevant material from [7].

**Definition 1.** A function \( \psi : [0, \infty) \to [0, \infty) \) is said to be an altering distance function if \( \psi \) is continuous and strictly increasing and if \( \psi(t) = 0 \) if and only if \( t = 0 \).

Follow the notation used in [7], \( \Phi \) denotes the next set.

\[
\Phi = \left\{ \varphi : [0, \infty)^2 \to [0, \infty) \mid \varphi(0, 0) \geq 0, \; \varphi(x, y) > 0 \text{ if } (x, y) \neq (0, 0), \right. \\
\left. \varphi\left(\liminf_{n \to \infty} a_n, \liminf_{n \to \infty} b_n\right) \leq \liminf_{n \to \infty} \varphi(a_n, b_n) \right\}
\]

The next result is stated and proved in [7].

**Theorem 4.** Let \((X, d, s \geq 1)\) be a complete \( b \)-metric space and \( T, f \) self-mappings on \( X \) which satisfy

\[
\psi\left(\frac{sd(Tx, fy)}{s+1}\right) \leq 1 + \varphi\left(d(x, fy), d(y, Tx)\right),
\]

for all \( x, y \in X \), where \( \psi \) is an altering distance function, \( \varphi \in \Phi \) and \( T \) is continuous. Then \( T \) and \( f \) have a unique common fixed point.

**Remark 7.** We will now show how to dispense with the assumption on function \( \varphi \). Indeed, the condition (7) implies

\[
sd(Tx, fy) \leq \frac{d(x, fy)}{s+1} + \frac{d(y, Tx)}{s^3(s+1)},
\]

i.e.

\[
d(Tx, fy) \leq \frac{d(x, fy)}{s(s+1)} + \frac{d(y, Tx)}{s^4(s+1)} \\
\leq \frac{1}{s(s+1)}[d(x, fy) + d(y, Tx)].
\]

Let \( x_0 \in X \), \( x_1 = Tx_0 \) and \( x_2 = fx_1 \). Define the sequence \( \{x_n\} \) by \( x_{2n+1} = Tx_{2n} \) and \( x_{2n+2} = fx_{2n+1} \), for every \( n \geq 0 \). It follows that

\[
d(x_{2n+1}, x_{2n}) = d(Tx_{2n}, fx_{2n-1}) \\
\leq \frac{1}{s(s+1)}[d(x_{2n}, fx_{2n-1}) + d(x_{2n-1}, Tx_{2n})] \\
\leq \frac{1}{s(s+1)}[d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})] \\
\leq \frac{1}{s(s+1)}d(x_{2n-1}, x_{2n+1})
\]
\[
\leq \frac{1}{s+1}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})],
\]

and we obtain
\[
\left(1 - \frac{1}{s+1}\right) d(x_{2n+1}, x_{2n}) \leq \frac{1}{s+1} d(x_{2n-1}, x_{2n}),
\]
This clearly forces
\[
d(x_{2n+1}, x_{2n}) \leq \frac{1}{s} d(x_{2n-1}, x_{2n}).
\]

According to Lemma 1, we conclude that \(\{x_n\}\) is a \(b\)-Cauchy sequence.

In the notation of [21], \(\Psi\) stands for the family of all functions \(\psi, \varphi : [0, \infty) \to [0, \infty)\) with the properties:

(a) \(\varphi(t) < \psi(t)\) for each \(t > 0, \varphi(0) = \psi(0) = 0\);
(b) \(\varphi\) and \(\psi\) are continuous functions;
(c) \(\psi\) is increasing,

and \(\Theta\) denotes the set of all functions \(\theta : [0, \infty)^4 \to [0, \infty)\) satisfying the following conditions:

(a) \(\theta\) is continuous,
(b) \(\theta(p, q, r, s) = 0\) if and only if \(pqrs = 0\).

**Example 1.** The following functions belong to \(\Theta:\)

1) \(\theta(p, q, r, s) = k \min\{p, q, r, s\} + p \cdot q \cdot r \cdot s, \quad k > 0,\)
2) \(\theta(p, q, r, s) = \ln (1 + p \cdot q \cdot r \cdot s).\)

Also in [21], a partially ordered set in a \(b\)-metric space and a regular space were introduced.

**Definition 2** ([21] Definition 2.1). Let \(X\) be a nonempty set. Then \((X, d, \preceq)\) is called a partially ordered \(b\)-metric space if \(d\) is a \(b\)-metric on a partially ordered set \((X, \preceq)\). The space \((X, d, \preceq)\) is called regular if the following condition holds: if a non-decreasing sequence \(\{x_n\}\) tends to \(x\), then \(x_n \preceq x\) for all \(n\).

The next theorem is the main result in [21].

**Theorem 5.** Suppose that \((X, d, s \geq 1, \preceq)\) is a partially ordered complete \(b\)-metric space and \(\{T_n\}\) a nondecreasing sequence of self maps on \(X\). If there exists a continuous function \(\alpha : X \times X \to [0, 1)\) such that for all \(x, y \in X\)
\[
\alpha(T_i x, T_j y) \leq a_{i,j} \alpha(x, y)
\]
and
\[
\psi\left(s^3 d(T_i x, T_j y)\right) \leq \alpha(x, y) \varphi(M_{i,j}(x, y)) + \theta(d(x, T_i x), d(y, T_j y), d(x, T_j y), d(y, T_i x)),
\]

and we obtain
\[
\left(1 - \frac{1}{s+1}\right) d(x_{2n+1}, x_{2n}) \leq \frac{1}{s+1} d(x_{2n-1}, x_{2n}),
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\]

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\]

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\[
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\]

This clearly forces
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\]

According to Lemma 1, we conclude that \(\{x_n\}\) is a \(b\)-Cauchy sequence.
for all \(x, y \in X\) with \(x \preceq y\), where \((\psi, \varphi) \in \Psi, \theta \in \Theta\) and

\[
M_{i,j} (x, y) = \max \left\{ d(x, y), d(x, T_i x), d(y, T_j x), \frac{d(x, T_j y) + d(y, T_i x)}{2s} \right\},
\]

and \(0 \leq a_{i,j} \ (i, j \in \mathbb{N})\), satisfy

(i) \(A_n = \prod_{i=1}^{n} a_{i,i+1} < 1\), for all \(n\),

(ii) \(\lim_{i \to \infty} a_{i,j} < 1\), for each \(j\).

Suppose that:

(i) \(T\) is continuous, or

(ii) \((X, d, \preceq)\) is regular.

If there exists \(x_0 \in X\) such that \(x_0 \preceq T x_0\), then all \(T_n\)'s have a common fixed point in \(X\).

Remark 8. The proof of Theorem 5 can be much shorter using Lemma 1. Indeed, on page 59, the proof should start with \(\psi(s^3d(T_n(x_{n-1}), T_{n+1}(x_n)))\), and it follows that

\[
\psi(s^3d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
\leq \alpha (x_{n-1}, x_n) \varphi (M_{n,n+1} (x_{n-1}, x_n)) \\
+ \theta (d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\
= \alpha (x_{n-1}, x_n) \varphi (M_{n,n+1} (x_{n-1}, x_n)) \\
\leq A_{n-1} \alpha (x_0, x_1) \varphi (\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).
\]

If \(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_n, x_{n+1})\}\), then we have

\[
\psi(d(x_n, x_{n+1})) \leq \psi(s^3d(T_n(x_{n-1}), T_{n+1}(x_n))) \\
\leq A_{n-1} \alpha (x_0, x_1) \varphi (d(x_n, x_{n+1})) \\
\leq A_{n-1} \alpha (x_0, x_1) \psi(d(x_n, x_{n+1})) \\
< \psi(d(x_n, x_{n+1})),
\]

which is impossible.

We can conclude that \(\psi(s^3d(T_n(x_{n-1}), T_{n+1}(x_n))) \leq \psi(d(x_n, x_{n+1}))\). If \(x_n \neq x_{n+1}\) for all \(n \in \mathbb{N}\), then

\[
d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \quad \lambda \leq \frac{1}{s^3}.
\]

By Lemma 1, the sequence \(\{x_n\}\) is a \(b\)-Cauchy sequence and \(x_n \to x\) as \(n \to +\infty\).

In [13] the authors defined Chatterjea’s type contraction in the context of \(b\)-metric spaces and proved the following result.
Theorem 6 ([13] Theorem 2.1). Let \((X, d, s \geq 1)\) be a complete b-metric space, \(d\) a continuous function, \(T : X \to X\) a Chatterjea’s map such that the inequality \(\sup_{n \in \mathbb{N}} d(T^n x, x) < \infty\) holds for all \(x \in X\). Then
(i) there exists a unique fixed point (say \(\xi\)) of \(T\);
(ii) for any \(x_0 \in X\) the sequence \(\{x_n\}\) converges to \(\xi\), where \(x_{n+1} = Tx_n\), \(n = 0, 1, 2, \ldots\);
(iii) there holds the a priori error estimate

\[
d(\xi, T^m x) \leq \alpha^m \sup_{j \in \mathbb{N}} d(T^j x, x).
\]

Recently, C. Chifu and G. Petrusel ([17], Theorem 2.1, Theorem 2.2.) considered the existence of fixed points for some multi-valued mappings in the context of b-metric spaces.

Theorem 7 ([17] Theorem 2.1). Let \((X, d, s > 1)\) be a complete b-metric space and \(T : X \to P(X)\) a multi-valued operator such that:
(i) there exist \(a, b, c \in \mathbb{R}_+, \ a + b + 2cs < \frac{s-1}{s^2}\) and \(b + cs < \frac{1}{s}\) such that
\[
H(T(x), T(y)) \leq ad(x, y) + b[D(x, T(x)) + D(y, T(y))] + c[D(x, T(y)) + D(y, T(x))],
\]
for all \(x, y \in X\);
(ii) \(T\) is closed.
In these conditions \(\text{Fix}(T) \neq \emptyset\).

Theorem 8 ([17] Theorem 2.2). Let \((X, d, s > 1)\) be a complete b-metric space and \(T : X \to P(X)\) a multi-valued operator such that:
(i) there exist \(a, b, c \in \mathbb{R}_+, \ a + b + 2cs < \frac{s-1}{s^2}\) and \(b + cs < \frac{1}{s}\) such that
\[
H(T(x), T(y)) \leq ad(x, y) + b[D(x, T(x)) + D(y, T(y))] + c[D(x, T(y)) + D(y, T(x))],
\]
for all \(x, y \in X\);
(ii) \(T\) is closed.
If \(\text{SFix}(T) \neq \emptyset\), then \(\text{SFix}(T) = \text{Fix}(T) = \{x\}\).

Remark 9. Note that we did not really have to use the condition \(b + cs < \frac{1}{s}\). Indeed, since \(a + b + 2cs < \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2} < \frac{1}{s}\), then \(b + cs < a + b + 2cs < \frac{1}{s}\).

Remark 10. In Theorem 7 and 8, the contractive condition

\[
H(T(x), T(y)) \leq ad(x, y) + b[D(x, T(x)) + D(y, T(y))] + c[D(x, T(y)) + D(y, T(x))],
\]
where \( a + b + 2cs < \frac{s-1}{s^2} \), can be replaced by the next two conditions:

\[
H (T(x), T(y)) \leq a_1 d(x, y) + b_1 D(x, T(x)) + c_1 D(y, T(y)) + d_1 [D(x, T(y)) + D(y, T(x))],
\]

where \( a_1 + \frac{b_1 + c_1}{2} + 2d_1 s < \frac{s-1}{s^2} \) and

\[
H (T(x), T(y)) \leq \lambda_1 d(x, y) + \lambda_2 D(x, T(x)) + \lambda_3 D(y, T(y)) + \lambda_4 D(x, T(y)) + \lambda_5 D(y, T(x)),
\]

with \( \lambda_1 + \frac{\lambda_2 + \lambda_3}{2} + s(\lambda_4 + \lambda_5) < \frac{s-1}{s^2} \).

We will now show that all the above contractive conditions are equivalent to each other.

It is easily seen that (9) implies (10) and from (10) we have (11).

Substituting \( H (T(y), T(x)) \) into (11.1) and combining with (11), we obtain

\[
H (T(x), T(y)) \leq \lambda_1 d(x, y) + \frac{\lambda_2 + \lambda_3}{2} [D(x, T(x)) + D(y, T(y))] + \frac{\lambda_4 + \lambda_5}{2} [D(x, T(y)) + D(y, T(x))].
\]

When \( a = \lambda_1, b = \frac{\lambda_2 + \lambda_3}{2}, c = \frac{\lambda_4 + \lambda_5}{2} \), we have (9).

Each of them can be associated with the general conditions which are considered in the metric spaces:

\[
H (T(x), T(y)) \leq k_1 \max \left\{ d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2s}, \frac{D(x, Ty) + D(y, Tx)}{2s} \right\},
\]

where \( k_1 = a + 2bs + 2cs \), and

\[
H (T(x), T(y)) \leq k_2 \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s} \right\},
\]

with \( k_2 = a_1 + b_1 + c_1 + 2sd_1 \), and also

\[
H (T(x), T(y)) \leq k_3 \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s} \right\},
\]

where \( k_3 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \).

It follows easily that (12) implies (13) and (13) implies (14).
References


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