SOME CHARACTERIZATIONS OF DIFFERENTIABILITY
OF CONVEX FUNCTIONS ON SMALL SET

Consider a real-valued locally Lipschitz function $f$ defined on a nonempty closed convex set $C$ (possibly $\text{int } C = \emptyset$), with nonempty nonsupport points $N(C)$ of a Banach space $E$, the relationship of differentiability (subdifferentiability) between $f$ and the Minkowski gauge by $\text{epi } (f)$ and a characterization of Gateaux (Fréchet) differentiability of Minkowski gauge on a small set are given.

Let $E$ be a real Banach space, $C$ a closed convex subset of $E$ and $f$ a real-valued convex function defined on $C$. Rainwater [1] and Verona [13] generalized Asplund's theorem [14], resp., Mazur's theorem to closed convex set $C$ which may have empty interior. They did so by substituting the set $N(C)$ of non support points of set $\mathcal{C}$ to the interior of $C$. In § 1 of the present paper, an equivalence of the differentiability (subdifferentiability) between $f$ and the Minkowski gauge by $f$ is established, and, in § 2, a characterization of Gateaux (Fréchet) differentiability of $f$ is given.

Key words: Gateaux differentiability, Fréchet differentiability, real-valued locally Lipschitz function, convex function, subdifferential, convex set of Banach space.

First, recall a sequence of definitions.

**Definition**

a) A point $x \in C$ is called a support point of $C$ provided there exists a nonzero $x^* \in E^*$ such that

$$\langle x^*, x \rangle = \sup \{ \langle x^*, y \rangle ; y \in C \}$$

The set of all points in $C$ which are not support points is denoted by $N(C)$ [see [1] and [5] for the properties of $N(C)$].

b) If $x \in C$, we denote by $C_x$ the cone generated by $C$ from $x$, that is, $y \in C_x$ provided there exists some $t > 0$ such that $x + ty \in C$.

c) The subdifferential $\partial f(x)$ of the convex function $f$ at the point $x \in C$ is defined to be the set of all $x^* \in E^*$ satisfying

$$\langle x^*, y-x \rangle \leq f(y) - f(x) \text{ for all } y \in C.$$  

d) We say that $f$ is Gateaux differentiable at $x \in N(C)$ provided $\partial f(x)$ is single-valued, equivalently, provided there is a unique $x^* \in E^*$ such that

$$\langle x^*, y-x \rangle \leq f(y) - f(x) \text{ for all } y \in C.$$
e) We say that $f$ is Fréchet differentiable at $x \in N(C)$ provided that there exists $x^* \in E^*$ such that for any $\varepsilon > 0$, there is $\delta > 0$ such that both $y \in C$ and $\|x - y\| < \delta$ imply that

$$0 \leq f(y) - f(x) - \langle x^*, y - x \rangle \leq \varepsilon \|y - x\|.$$ 

The letter $C$ will always denote a closed convex subset of $E$ with $N(C) \neq \emptyset$, and we denote by $\text{dom}(f)$ the essential domain of $f$, that is, $\text{dom}(f) = \{x \in E; f(x) < \infty\}$.

1. In this section, we will see a close relationship of the differentiability (subdifferentiability) between $f$ and the Minkowski gauge by $f$.

**Theorem 1.1.** If $f$ is locally Lipschitzian on $N(C)$ and convex on $C$. Then for any $x_0 \in N(C)$ there exists Minkowski gauge $P$ by $\text{epi}(g)$ such that the subdifferential map $\partial f$ exists at $x \in N(C)$ if and only if the subdifferential map $\partial P$ exists at the point $(x - x_0, f(x) - r_0)$, where $g(x) = f(x - x_0) - r_0$, $r_0 = 1 + f(x_0)$ and $\text{epi}(g) = \{(y, t) \in E \times R, t \geq g(y)\}$.

**Proof.** We pair $E \times R$ with its dual $E^* \times R$ by

$$\langle (x^*, r^*), (x, r) \rangle = \langle x^*, x \rangle + r^* \cdot r$$

The definition of $g$ implies that $g$ is locally Lipschitzian and convex on the convex set $N(K)$, where $K = C - x_0$ with $0 \in N(K)$ and $g(0) = -1$. Thus, $\text{epi}(g)$ is closed and convex and contains the origin of $E \times R$.

Suppose that the subdifferential map $\partial P$ of $P$ exists at $(u, g(u)) \in \text{epi}(g)$ with $u \in N(K)$, choose a subdifferential of $P$ at $(u, g(u))$, $(u^*, r^*)$, say, that is

$$\langle (u^*, r^*), (y, r) - (u, g(u)) \rangle \leq P(y, r) - P(u, y(u))$$

for all $(y, r) \in \text{dom}(P)$, or equivalently,

$$(*) \quad \langle u^*, y - u \rangle + r^*(r - y(u)) \leq p(y, r) - 1, \ (y, r) \in \text{dom}(P).$$

Note that for each $y \in K$, $P(y, r) \leq 1$ whenever $r \geq g(y)$, this and $(*)$ imply $r^* \leq 0$ by letting $r$ which tends to the positive intinity, and further we have $r^* < 0$. Suppose, to the contrary, that $r^* = 0$, then either $u^* \neq 0$, in which case $\langle u^*, y - u \rangle \leq 0$ for all $y \in K$ by taking $r = g(y)$, a contradiction to $u \in N(K)$; or $u^* = 0$, in which case we have $p(y, r) \geq 1$ for all $(y, r) \in \text{dom}(P)$, a contradiction to $P(0, 0) = 0$. Since $P(y, g(y)) = 1$, by taking $r = g(y)$ in $(*)$ we have
\[ \langle u^*, y - u \rangle \leq -r^*(g(y) - g(u)) \] for all \( y \in K \)

that is, \(-1/r^*)u^* \in \partial g(u)\). Hence the theorem is shown in one direction.

Conversely, suppose that the subdifferential map \( \partial g \) exists at \( u \in N(K) \) and let \( u^* \in \partial g(u) \), that is,

\[ \langle u^*, y - u \rangle \leq g(y) - g(u) \] for all \( y \in K \)

Taking \( y = 0 \), we see that \( \langle u^*, u \rangle - g(u) \geq 1 \), define \( s(r) = r/(\langle u^*, u \rangle - g(u)) \) for all \( r \in \mathbb{R} \). By an argumentation which is close to the one given in [2, p13] we know that \( s(1) (u^*, -1) \in E^* \times \mathbb{R} \) is a subdifferential to \( P \) at the point \((u, g(u))\).

**Proposition 1.2.** Suppose that \( f \) is convex on \( C \), then \( f \) is locally Lipschitzian on \( N(C) \) if and only if \( \partial f \) is locally bounded on \( N(C) \).

**Proof.** The necessity follows immediately from [1].

Sufficiency. Suppose that \( \partial P \) is locally bounded on \( N(C) \), that is, for any \( z \in N(C) \), there is a neighbourhood \( U \) of \( z \) in \( N(C) \) such that

\[ M = 3 \sup \{ \| x^* \| ; \ x^* \in \partial f(x), \ x \in U \} < \infty . \]

For any \( x, y \in U \) and for all \( x^* \in \partial f(x) \) and \( y^* \in \partial f(y) \), we have

\[ \langle y^*, y - x \rangle \geq f(y) - f(x) \geq \langle x^*, y - x \rangle . \]

This implies

\[ \frac{2}{3} M \| y - x \| \geq | \langle y^* - x^*, y - x \rangle | \geq | f(y) - f(x) - \langle x^*, y - x \rangle | \geq \]

\[ \geq | f(y) - f(x) | - | \langle x^*, y - x \rangle | \geq f(y) - f(x) | - \frac{M}{3} \| y - x \|. \]

Thus we have

\[ | f(y) - f(x) | \leq M \| y - x \| \] for all \( x, y \in U \)

and this explains that \( f \) is locally Lipschitzian on \( N(C) \).

**Lemma 1.3.** Suppose that \( f \) is locally Lipschitzian on \( N(C) \) and convex on \( C \) and suppose that \( 0 \in N(C) \) with \( f(0) = -1 \). Then the Minkowski gauge \( P \) by \( epi(f) \) is also locally Lipschitzian on \( N(dom(P)) \).
Proof. It is easy to observe that for any \((x, r) \in N(\text{dom}(P))\) there is \(\lambda > 0\) such that \(\lambda x \in N(C)\). By positive homogeneity of \(P\) and the proof of Proposition 1.2, it suffices to show that there is a selection \(\phi\) for \(\partial P\) on \(N(\text{dom}(P))\) which is locally bounded on \(\{(x, f(x)); x \in N(C)\} = \text{graph}_{N(C)} f\).

Let \(\psi\) be any selection for \(\partial f\) on \(N(C)\), then \(\psi\) is locally bounded on \(N(C)\) \([1]\), since \(f\) is locally Lipschitzian. For any \(x \in N(C)\), by the proof of the necessity of Theorem 1.1, \(\phi(x, f(x)) = S(x, 1) \cdot (\psi(x), -1) \in \partial P(x, f(x))\), where \(S(x, 1) = 1/(\langle \psi(x), x \rangle - f(x)) \leq 1\). Clearly, the local boundedness of \(\psi\) on \(N(C)\) implies the local boundedness of \(\phi\) on \(\text{graph}_{N(C)} f\) and which completes our proof.

Theorem 1.4. If \(f\) is locally Lipschitzian on \(N(C)\) and convex on \(C\), then for any \(x_0 \in N(C)\) there exists a Minkowski gauge \(P\) by \(\text{epi}(g)\) such that \(f\) is Gateaux differentiable at \(x \in N(C)\) if and only if \(P\) is Gateaux differentiable at \((x - x_0, f(x) - r_0)\), where \(g(y) = f(y + x_0) - r_0\) for all \(y \in C - x_0\) and \(r_0 = 1 + f(x_0)\).

Proof. Suppose that \(f\) is Gateaux differentiable at \(x \in N(C)\), equivalently, \(g\) is Gateaux differentiable at \(u = x - x_0 \in N(K)\) (where \(K = C - x_0\)), suppose that \((u_j^+, r_j^+), (u_j^-, r_j^-)\) for \(j = 1, 2\) are two subdifferentials to \(P\) at the point \((u, g(u))\), by using the similar argument of Theorem 1.1, we see that \(r_1^+ < 0\) for \(j = 1, 2\) and that \(- (1/r_j^+) \cdot u_j^+ \in \partial g(u)\) for \(j = 1, 2\). By the uniqueness of \(\partial g(u)\), we have

\[
(A) - \frac{1}{r_1^+}(u_1^+, r_1^+) = - \frac{1}{r_2^+}(u_2^+, r_2^+).
\]

Since \(\langle u_j^+, r_j^+ \rangle, (u, g(u)) \rangle = P(u, g(u)) = 1\) for \(j = 1, 2\), combining this and \((A)\), we obtain that \(r_1^+ = r_2^+\) which in turn implies \((u_1^+, r_1^+) = (u_2^+, r_2^+)\). It explains that \(\partial P\) is unique at \((u, g(u))\) and hence \(P\) is Gateaux differentiable at \((u, g(u))\).

Conversely, suppose that \(u_j^+\) for \(j = 1, 2\) are distinct subdifferentials to \(g\) at \(u\). Let \(s_j(r) = r/(\langle u_j^+, u \rangle - g(u))\) for \(j = 1, 2\) and \(r \in R\), again by an argument similar to \([1, P13]\), we obtain that \(s_j(1)(u_j^+, -1) \in \partial P(u, g(u))\) for \(j = 1, 2\). It is evident that \(s_1(1)(u_1^+, -1) \neq s_2(1)(u_2^+, -1)\) if \(u_1^+ = u_2^+\) and which established Theorem 1.4.

Theorem 1.5. If \(f\) is convex on \(C\) and locally Lipschitzian on \(N(C)\) then for any \(x_0 \in N(C)\) there is Minkowski gauge \(P\) by \(\text{epi}(g)\) such that \(f\) is Fréchet differentiable at \(x \in N(C)\) if and only if \(P\) is Fréchet differentiable at \((x - x_0, f(x) - r_0)\), where \(g(x) = f(x + x_0) - r_0\) and \(r_0 = 1 + f(x_0)\).
Proof. Clearly, \( N(\text{epi}(g)) \neq \emptyset \) (containing, say, the origin of \( E \times R \)) and by Lemma 1.3, \( P \) is locally Lipschitzian on \( \text{dom}(P) \). By [1], the subdifferential map \( \partial g(\partial P) \) exists on \( N(K) [N(\text{dom}(P))]. \) Suppose that \( g \) is Fréchet differentiable at \( u \in N(K), \) by the uniqueness of \( \partial g(u), \) we can assume that \( u^* = \partial g(u) \in E^*, \) and there is a selection \( \phi \) for the subdifferential map \( \partial g \) on \( N(K) \) with \( \psi(u) = u^* \) which is norm-to-norm continuous at \( u \) [1]. From the proof of necessity of

Theorem 1.1, for each such \( \phi(y), \) there exists \( r(y) \left[ \frac{1}{\langle \phi(y), y \rangle - g(y)} \right] \) with \( 0 < r(y) \leq 1 \) such that \( r(y) (\phi(y), -1) \in \partial P(y, g(y)). \) Thus \( \psi(y) = r(y) (\phi(y), -1) \) is a selection for \( \partial P \) on \( N(\text{dom}(P)). \) Obviously, both \( y \to u \) and \( \phi(y) \to \phi(u) \) imply that \( \psi(y) \to \psi(u), \) that is, there is a selection for \( \partial P \) on \( N(\text{dom}(P)) \) which is norm-to-norm continuous at \( (u, g(u)) \) and hence \( P \) is Fréchet differentiable at \( (u, g(u)) \) [1].

Conversely, if \( P \) is Fréchet differentiable at \( (u, g(u)), \) then there is a selection \( \psi \) for \( \partial P \) on \( N(\text{dom}(P)), \) of course, on \( \{y, g(y)\}; \ y \in N(K)\}, \) which is norm-to-norm continuous at \( (u, g(u)). \) Let \( \psi(y, g(y)) = (y^*, s^*) \) and let \( \psi(u, g(u)) = \partial P(u, g(u)) = (u^*, r^*). \) By Theorem 1.1 we have \( s^* < 0 \) and \( r^* < 0, \) for any \( y \in N(K). \) By the continuity of \( \psi, \) we have \( y^* \to u^*, \ s^* \to r^* \) whenever \( y \to u \) in \( N(K). \) Equivalently, \( -\frac{1}{s^*} y^* \to -\frac{1}{r^*} u^* \) whenever \( y \to u \) in \( N(K). \) Note that there is a selection \( \psi \) with \( \psi(y) = -\frac{1}{s^*} y^* \) for \( \partial g \) on \( N(K) \) which is norm-to-norm continuous at \( u \in N(K). \) Thus, \( g \) is Fréchet differentiable at \( u. \)

Remark. Theorems 1.1 and 1.4 still hold if \( f \) is assumed to be lower semi continuous and real valued on \( C. \)

2. Now, we consider the differentiability of Minkowski gauge by a closed convex set \( C. \)

Theorem 2.1. Let \( P \) be a Minkowski gauge on \( E \) by \( C \) with \( 0 \in N(C). \) Suppose that \( P \) is locally Lipschitzian on \( N(C), \) then for \( x \in N(C) [P(x) > 0] \) it is Gateaux differentiable at \( x \) if and only if for each finite dimensional subspace \( F \subset E \) which contains \( x \) there holds

\[
\lim_{r \to 0^+} \sup_{u, v \in S(x, P) \cap U_F(x, r)} \frac{P(x) - P\left(\frac{u + v}{2}\right)}{\|u - v\|} = 0
\]

(a)
where \( U_F(x, r) = U(x, r) \cap F, U(x, r) = \{ y \in E; ||x - y|| < r \} \) and \( S(x, P) \) denotes the level set \( \{ y \in E; P(y) = P(x) \} \).

If \( P(x) = 1 \), we shall also write the level set \( S \) instead of \( S(x, P) \) for fixed \( x \in E \).

**Proof.** Since \( P \) is locally Lipschitzian on \( N(C) \), by [1], \( \partial P \) exists on \( N(\text{dom}(P)) \). By positive homogeneity we can assume that \( P(x) = 1 \).

Suppose that \( P \) is not Gateaux differentiable at \( x \) and suppose that \( x^+_j \) for \( j = 1, 2 \), are two distinct subdifferentials to \( P \) at \( x \), then we have

\[
\langle x^+_j, x \rangle = p(x) = 1 \quad \text{for} \quad j = 1, 2.
\]

Let \( h = \frac{x^+_1 + x^+_2}{2} \), one can choose \( z \in h^{-1}_0 \) with \( ||z|| = 1 \) such that

\[
\langle x^+_j, z \rangle = \langle -x^+_j, z \rangle = 2a > 0 \quad \text{for some} \quad a > 0.
\]

Note that \( - C_x \) is also dense in \( E \), since \( C_x \) is dense in \( E \), one can also choose \( y_1 \in C_x \) and \( y_2 \in -C_x \) with \( \|y_j\| = 1 \) for \( j = 1, 2 \) and with \( \|y_1 - y_2\| \leq a/M \) such that

\[
\langle x^+_1, y_1 \rangle \geq a \quad \text{and} \quad \langle x^+_2, -y_2 \rangle \geq a,
\]

where \( M \) is Lipschitz constant of \( P \) at \( x \). Thus, for sufficiently small \( t > 0 \), we have

\[
x + ty_1 \in C \quad \text{and} \quad x - ty_2 \in C.
\]

So we obtain that

\[
P(x + ty_1) - P(x) \geq \langle x^+_1, ty_1 \rangle \geq at
\]

and

\[
P(x - ty_2) - P(x) \geq \langle x^+_2, -ty_2 \rangle \geq at.
\]

For each such \( t \), set \( \alpha_t = 1/P(x + ty_1), \beta_t = 1/P(x - ty_2) \) and set \( u_t = \alpha_t(x + ty_1) \) and \( v_t = \beta_t(x - ty_2) \). Then \( u_t, v_t \in S \) and

\[
|\alpha_t - \beta_t| \leq |P(x + ty_1) - P(x - ty_2)|/(1 + at)^2 \leq
\]

\[
\leq |P(x + ty_1) - P(x - ty_2)| \leq Mt\|y_1 + y_2\| \leq 2Mt,
\]

\[
\|u_t - v_t\| = ||(\alpha_t - \beta_t)x + (\alpha_t y_1 + \beta_t y_2)t|| \leq
\]

\[
\leq |\alpha_t - \beta_t| ||x|| + (\alpha_t + \beta_t) t \leq (2M||x|| + 2)t = Lt,
\]
\[ P\left(\frac{u_t+v_t}{2}\right) = P\left(\frac{\alpha_t + \beta_t}{2} x + \frac{1}{2} t(\alpha_t y_1 + \beta_t y_2)\right) \leq P\left(\frac{\alpha_t + \beta_t}{2} x\right) + M \left\| t(\alpha_t y_1 - \beta_t y_2) \right\| \leq \frac{\alpha_t + \beta_t}{2} + \frac{M \cdot t}{2} \left[ |\alpha_t - \beta_t| + |\beta_t| \left\| y_1 - y_2 \right\| \right] \leq \frac{1}{1+at} + (Mt)^2 + \frac{a}{2} t. \]

Thus

\[ \frac{1-P\left(\frac{u_t+v_t}{2}\right)}{||u_t-v_t||} \geq \frac{1-\left(\frac{1}{1+at} + (Mt)^2 + \frac{a}{2} t\right)}{L \cdot t} \rightarrow \frac{a}{2L} > 0 \]

whenever \( t \to 0^+ \). By taking \( F = \text{span}\{x, y_j; \text{ for } j = 1, 2\} \) we have

\[ \lim_{r \to 0^+} \sup_{u,v \in S \cap U_F(x,r)} \frac{P(u) - P\left(\frac{u+v}{2}\right)}{||u-v||} \geq \lim_{r \to 0^+} \frac{1-P\left(\frac{u_t+v_t}{2}\right)}{||u_t-v_t||} \geq \frac{a}{2L} > 0. \]

Conversely, suppose that \( P \) is Gateaux differentiable at \( x \). We can assume, again as before, that \( P(x) = 1 \). By [1] there is a selection \( \psi \) for \( \partial P \) on \( N(C) \) which is norm-to-weak* continuous at \( x \). For every finite dimensional subspace \( F \) of \( E \) which contains \( x \) and for \( u, v \in S \cap F \) with \( u \neq v \), we have

\[ 0 \leq \frac{1-P\left(\frac{u+v}{2}\right)}{||u-v||} = \frac{P(u) - P\left(\frac{u+v}{2}\right)}{||u-v||} = \]

\[ = \frac{P(u) - P\left(\frac{u+t v-u}{||v-u||}\right)}{2t} \left[ \text{where } t = \frac{1}{2} ||v-u|| \right] \leq \frac{1}{2} \langle \psi(u), \frac{v-u}{||v-u||} \rangle = \]

\[ = \frac{1}{2} \langle \psi(u) - \psi(v), \frac{v-u}{||v-u||} \rangle - \frac{1}{2||v-u||} \left[ \langle \psi(v), v \rangle - \langle \psi(u), u \rangle \right] \leq \]

\[ \leq \frac{1}{2} \langle \psi(u) - \psi(v), \frac{u-v}{||u-v||} \rangle \rightarrow 0 \]

whenever \( u, v \in S \cap F, u \neq v \) and \( u, v \rightarrow x \), i.e.
\[
\lim_{r \to 0^+} \sup_{u, v \in \mathcal{S} \cap \mathcal{U}(x, r) \atop u \neq v} \frac{P(x) - P\left(\frac{u + v}{2}\right)}{||u - v||} = 0.
\]

**Theorem 2.2.** With the same assumptions as in Theorem 2.1 on the set \(\mathcal{C}\), the function \(P\) and the point \(x\), if \(P\) Fréchet differentiable at \(x\), then we have

\[
\lim_{r \to 0^+} \sup_{u, v \in \mathcal{S} \cap \mathcal{U}(x, r) \atop u \neq v} \frac{P(x) - P\left(\frac{u + v}{2}\right)}{||u - v||} = 0
\]

(b)

if, in addition, \(\text{int} \mathcal{C} \neq \emptyset\), then \(P\) is Fréchet differentiable at \(x\) if and only if (b) holds.

**Proof.** Apply Fréchet differentiability of \(P\) at \(x\) in place of Gateaux differentiability in the proof of necessity of theorem 2.1 and recall that \(P\) is Fréchet differentiable at \(x\) if and only if there is a selection for the subdifferential map \(\partial P\) which is norm-to-norm continuous at \(x\), so the first part of this theorem is proved, immediately.

If \(\text{int} \mathcal{C} = \emptyset\), then \(\text{int} \mathcal{C} = N(C) [1]\). Let (b) hold. Then \(P\) is Gateaux differentiable at \(x\). Suppose that \(\mathcal{F}\) is not Fréchet differentiable at \(x\), then there exists \(\varepsilon > 0\) and sequences \(\{x_n\} \subset E \cap x^*-1(0)\) with \(||x_n|| = 1\) and \(\{t_n\} \subset R^+, t_n \to 0\) such that

\[
\frac{P(x + t_n x_n) - P(x)}{t_n} > \varepsilon \quad \text{where} \quad x^* = \partial P(x).
\]

It is easy to see that

\[
P(x + t_n x_n) \geq 1 + t_n \varepsilon \quad \text{and} \quad P(x - t_n x_n) \geq 1
\]

for all sufficiently large \(n\) such that \(x \pm t_n x_n \in \mathcal{C}\).

Let \(\alpha_n = 1/P(x + t_n x_n), \beta_n = 1/P(x - t_n x_n), u_n = \alpha_n (x + t_n x_n)\) and \(v_n = \beta_n (x - t_n x_n)\). Then \(u_n, v_n \in \mathcal{S}\). Now we have

\[
|\alpha_n - \beta_n| \leq |x_n - 1| + |1 - \beta_n| \leq 2M t_n
\]

(\(M\) is locally Lipschitz constant of \(P\) at \(x\)),

\[
||u_n - v_n|| = ||(\alpha_n - \beta_n) x + (\alpha_n + \beta_n) t_n x_n|| \leq 2M ||x|| t_n + 2||x_n|| t_n = 2M ||x|| (1 + 1) t_n,
\]
\[
P\left(\frac{u_n + v_n}{2}\right) = P\left(\frac{\alpha_n + \beta_n}{2} x + \frac{1}{2}(\alpha_n - \beta_n) t_n x_n\right) \leq P\left(\frac{\alpha_n + \beta_n}{2} x\right) + M \left\|\frac{1}{2}(\alpha_n - \beta_n) t_n x_n\right\| \leq
\]
\[
\leq \frac{\alpha_n + \beta_n}{2} + M^2 t_n^2 \|x_n\| \leq \frac{1}{2} + \frac{1}{2(1 + \varepsilon t_n)} + M^2 t_n^2,
\]
\[
1 - P\left(\frac{u_n + v_n}{2}\right) \geq \frac{1}{2} - \frac{1}{2(1 + \varepsilon t_n)} - M^2 t_n^2 = \frac{\varepsilon^* t_n}{2(1 + \varepsilon^* t_n)} - M^2 t_n^2.
\]

Thus
\[
\lim_{r \to 0+} \sup_{\|u - v\| \leq 1} \frac{1 - P\left(\frac{u + v}{2}\right)}{\|u - v\|} \geq \lim_{n \to \infty} \frac{1 - P\left(\frac{u_n + v_n}{2}\right)}{\|u_n - v_n\|} \geq \frac{\varepsilon}{4(M\|x\| + 1)}.
\]

This is a contradiction which completes our proof.

We know that the set of Fréchet differentiability points of locally Lipschitz convex function \(f\) defined on nonempty, open convex set \(D\) is always \(G_\varepsilon\) — set of \(D\). Rainwater [1] proved that if \(N(C) \neq \emptyset\), then the above conclusion still holds in Asplund space. Wu and Cheng [6] pointed out if \(N(C) \neq \emptyset\) then the set of Gateaux differentiability points is always a \(G_\delta\) — set of \(N(C)\) in weak Asplund space. For any Banach space \(E\), we have

**Theorem 2.3.** Let \(f\) be convex \(C\) and locally Lipschitzian on \(N(C)\). Then a set \(G \subset N(C)\) on which \(f\) is Fréchet differentiable is contained in some \(G_\delta\) — set of \(N(C)\) on which \(f\) is Gateaux differentiable.

**Proof:** Without loss of generality we assume that \(0 \in N(C)\) and \(f(0) = -1\). Let \(P\) be the Minkowski functional by epi \((f)\). Since \(f\) is locally Lipschitzian on \(N(C)\), by Lemma 1.3, \(P\) is locally Lipschitzian on \(N(\text{dom}(P))\). By Theorem 1.4 and Theorem 1.5, \(f\) is Gateaux (Fréchet) differentiable at \(x \in N(C)\) if and only if \(P\) is Fréchet differentiable at \((x, f(x))\). Therefore, it suffices to show that the set \(Q \subset \text{graph}(f) \cap N(\text{dom}(P))\) on which \(P\) is Fréchet differentiable is contained in some \(G_\delta\) — set of \(N(C)\) of Gateaux differentiability points.

Let \(\text{graph}(f) = S\). For each integer \(n \geq 1\), let

\[
G_n = \left\{ x \in N(\text{dom}(P)) \cap S; u, v \in S \cap U(x, r) \sup_{u \neq v} \frac{1 - P\left(\frac{u + v}{2}\right)}{\|u - v\|} < \frac{1}{n}, \text{ for some } r > Q \right\}
\]
and let

\[ G = \bigcap G_n. \]

By Theorems 2.1 and 2.2, \( Q \subset G \) and \( f \) is Gateaux differentiable on \( G \). It remains to show that \( G_n \) is open set of \( S \) for every \( n \). For any \( x \in G_n \), let \( r(>0) \) satisfy

\[ \sup_{u,v \in S} \frac{P(x) - P\left(\frac{u + v}{2}\right)}{\|u - v\|} < \frac{1}{n}. \]

Note that \( S \cap U(\frac{r}{2}) \subset S \cap U(x, r) \) whenever \( y \in S \cap U\left(\frac{r}{2}\right) \), so we have

\[ \sup_{u,v \in S} \frac{P(y) - P\left(\frac{u + v}{2}\right)}{\|u - v\|} \leq \sup_{u,v \in S} \frac{P(x) - P\left(\frac{u + v}{2}\right)}{\|u - v\|} < \frac{1}{n} \]

whenever \( y \in S \cap U\left(\frac{r}{2}\right) \), i.e. \( S \cap U\left(\frac{r}{2}\right) \subset G_n \). Since \( S \cap U\left(\frac{r}{2}\right) \) is open subset of \( S \), this says that \( G \) is \( G_\delta \) set of \( S \).

References


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