COMPARISON THEOREMS FOR DIFFERENCE EQUATIONS

In the paper some comparison theorems for nonlinear difference equations of higher orders are presented.

Key words: difference equation, initial conditions, comparison theorems.

The comparison theorems play a very important role in the theory of difference equations as well as in the theory of differential equations. For the case of 2-nd order difference equations the problem was considered by Hooker, Patula [2], Olver [3], Patual [4]. Comparison theorems for difference equations of higher orders were considered by Popenda [5]. See also to the recent monograph by Agarwal [1] and the papers quoted there.

In this paper we present the comparison theorems for some types of nonlinear difference equations of an arbitrary order.

Let $N$ denote the set of positive integers, $R$ the set of real numbers and $R_+$ the set of positive real numbers. For a given function $x : N \to R$ we define the difference operators $\Delta^i$ as follows

$$\Delta^o x_n = x_n, \quad \Delta^k x_n = \Delta(\Delta^{k-1} x_n) = \Delta^{k-1} x_{n+1} - \Delta^{k-1} x_n, \quad k \geq 1$$

where $x_n = x(n)$.

Theorem 1. Let $y$ and $z$ be respectively the solutions of equations

(E1) \quad $\Delta^m y_n + f(n, y_n) = 0$, \quad $n \in N$

(E2) \quad $\Delta^m z_n + g(n, z_n) = 0$, \quad $n \in N$

where $f, g : N \times R \to R_+$. If $g(n, x) \geq f(n, x)$ for all $n \geq 1$ and $x \in R$, $f$ or $g$ is a nondecreasing function with respect to $x$ for each $n \in N$, and if $y$ and $z$ satisfy the conditions

(1) \quad $\Delta^\mu (y_1 - z_1) \geq 0$, \quad $\mu = 0, 1, \ldots, m-1$, 
then

(2) \[ \Delta^\mu(y_{n+1} - z_{n+1}) \geq \Delta^\mu(y_n - z_n) \geq 0, \quad n \geq 1, \quad \mu = 0, 1, \ldots, m - 1. \]

Proof. In the proof we will apply the mathematical induction. Let \( n = 1 \). From (1) we have \( \Delta^\mu(y_1 - z_1) \geq 0 \) for \( \mu = 0, 1, \ldots, m - 1 \). Therefore

\[ \Delta^{\mu-1}(y_2 - z_2) \geq \Delta^{\mu-1}(y_1 - z_1) \geq 0, \quad \mu = 1, \ldots, m - 1, \]

that is \( \Delta^\mu(y_2 - z_2) \geq \Delta^\mu(y_1 - z_1) \geq 0, \quad \mu = 1, \ldots, m - 2 \).

We will show that \( \Delta^{m-1}(y_2 - z_2) \geq \Delta^{m-1}(y_1 - z_1) \geq 0 \). Subtracting the members of equations (E1) and (E2) we get the equality

(3) \[ \Delta^m(y_n - z_n) = g(n, z_n) - f(n, y_n). \]

Therefore for \( n = 1 \) we have

\[ \Delta^m(y_1 - z_1) = g(1, z_1) - f(1, y_1). \]

But for \( \mu = 0 \) (1) implies that \( y_1 \geq z_1 \), the function \( f \) or \( g \) is nonnegative and nondecreasing, hence

\[ \Delta^m(y_1 - z_1) \geq g(1, z_1) - f(1, y_1) \geq 0 \]

or

\[ \Delta^m(y_1 - z_1) \geq g(1, y_1) - f(1, y_1) \geq 0. \]

Therefore

\[ \Delta^{m-1}(y_2 - z_2) \geq \Delta^{m-1}(y_1 - z_1) \geq 0. \]

Now, let us assume that the inequality (2) holds for \( n = k \), i.e.

(4) \[ \Delta^\mu(y_{k+1} - z_{k+1}) \geq \Delta^\mu(y_k - z_k) \geq 0, \quad \mu = 0, 1, \ldots, m - 1. \]

Then

\[ \Delta^{\mu-1}(y_{k+2} - z_{k+2}) - \Delta^{\mu-1}(y_{k+1} - z_{k+1}) \geq 0, \quad \mu = 1, \ldots, m - 1. \]

Hence

\[ \Delta^\mu(y_{k+2} - z_{k+2}) \geq \Delta^\mu(y_{k+1} - z_{k+1}) \geq 0, \quad \mu = 0, 1, \ldots, m - 2. \]

For \( n = k + 1 \) we get from (3)

\[ \Delta^m(y_{k+1} - z_{k+1}) \geq g(k+1, z_{k+1}) - f(k+1, y_{k+1}). \]
From assumption (4) for \( \mu = 0 \) we obtain \( y_{k+1} \geq z_{k+1} \).

Hence, by virtue of the theorem's assumptions we get

\[
g(k+1, z_{k+1}) - f(k+1, y_{k+1}) \geq g(k+1, z_{k+1}) - f(k+1, z_{k+1}) \geq 0
\]
or

\[
g(k+1, z_{k+1}) - f(k+1, y_{k+1}) \geq g(k+1, y_{k+1}) - f(k+1, y_{k+1}) \geq 0.
\]

Therefore

\[
\Delta^{-1}(y_{k+2} - z_{k+2}) \geq \Delta^{-1}(y_{k+1} - z_{k+1}).
\]

Thus inequality (2) is true for all \( n \geq 1 \). This completes the proof of Theorem 1.

Remark. If we put \( f \equiv 0 \) in equation (E1) then the theorem given above can be used for estimation of solutions of equation (E2) by the polynomial functions.

Example. Let us consider the following equations

\[
\Delta^3 y_n = 0 \tag{e1}
\]

and

\[
\Delta^3 z_n + nz_n^2 = 0. \tag{e2}
\]

Let \( y_1 = z_1 = 1, \Delta y_1 = \Delta z_1 = 3, 2 = \Delta^2 y_1 \geq \Delta^2 z_1 = 0 \) be the initial conditions. It is easy to verify that the solution of equation (e1), satisfying the above initial conditions is a polynomial \( y = n^2 \). By virtue of Theorem 1 the solution of equation (e2), defined by the given initial conditions, has the estimation \( z_n \leq n^2 \) for all \( n \in \mathbb{N} \).

Theorem 2. Let the assumptions of Theorem 1 hold. Then we have

\[
\Delta^\mu(y_n - z_n) \geq \sum_{i=0}^{m-\mu} \binom{n-1}{i} \Delta^{i+\mu}(y_1 - z_1), \quad n \geq 1,
\]

\[
\mu = 0, 1, ..., m-1.
\]

Proof. From Theorem 1 (for \( \mu = 0 \)) it follows the sequence \( \{\Delta^{-1}(y_n - z_n)\}_{n=1}^{\infty} \) is nondecreasing. Therefore

\[
\Delta^{-1}(y_n - z_n) \geq \Delta^{-1}(y_1 - z_1).
\]
Summing these inequalities over $n$ we get

$$\sum_{j=1}^{n-1} A^{m-1}(y_j - z_j) \geq \sum_{j=1}^{n-1} A^{m-1}(y_1 - z_1).$$

Hence

$$A^{m-2}(y_n - z_n) \geq A^{m-2}(y_1 - z_1) + (n - 1) A^{m-1}(y_1 - z_1).$$

Again by summation over $n$ we obtain

$$\sum_{j=1}^{n-1} A^{m-2}(y_j - z_j) \geq \sum_{j=1}^{n-1} A^{m-2}(y_1 - z_1) + \sum_{j=1}^{n-1} (j - 1) A^{m-1}(y_1 - z_1).$$

Hence

$$A^{m-3}(y_n - z_n) \geq A^{m-3}(y_1 - z_1) + (n - 1) A^{m-2}(y_1 - z_1) + \left( \sum_{j=1}^{n-1} \binom{j-1}{1} \right) A^{m-1}(y_1 - z_1).$$

Since $\sum_{j=1}^{n-1} \binom{j-1}{k} = \binom{n-1}{k+1}$ we have

$$A^{m-3}(y_n - z_n) \geq A^{m-3}(y_1 - z_1) + \binom{n-1}{1} A^{m-2}(y_1 - z_1) + \binom{n-1}{2} A^{m-1}(y_1 - z_1).$$

After $k$ steps we obtain

$$A^{m-k-1}(y_n - z_n) \geq \sum_{i=0}^{k} \binom{n-1}{i} A^{m-k-1+i}(y_1 - z_1), \quad k = 0, 1, \ldots, m - 1.$$  

Hence if we denote $\mu = m - k - 1$, then we obtain (5) and this completes the proof of Theorem 2.

**Corollary.** Applying (1) to inequalities (5) we obtain the estimations

$$A^{n}(y_n - z_n) \geq \binom{n-1}{m-\mu-1} A^{m-1}(y_1 - z_1), \quad \mu = 0, 1, \ldots, m - 1.$$

**Example.** Let $f \equiv 0$. Then the equations (E1) and (E2) have the form

(e1) \quad $A^{m}y_n = 0$

and

(e2) \quad $A^{m}z_n + g(n, z_n) = 0$. 

Let the initial conditions satisfy the relations \( \Delta^i z_1 \leq (m-1)^{(m-1-i)} (m-1)^{(m-1-i)} \) for \( i = 0, 1, \ldots, m-2 \), \( \Delta^{m-1} z_1 = 0 \).

The solution of equation (E1) satisfying conditions 
\( \Delta^i y_1 = (m-1)^{(m-1-i)} (m-1)^{(m-1-i)} \) for \( i = 0, 1, \ldots, m-1 \) is a polynomial 
\( y_n = (n+m-2)^{(m-1)} \). By virtue of Corollary (for \( \mu = 0 \)) the solution of equation (E2) defined by the given initial conditions has the estimation 
\( z_n \leq (n+m-2)^{(m-1)} - (n-1)^{(m-1)} \) for all \( n \in \mathbb{N} \).

**Remark.** If in equations (E1) and (E2) we will put \( f = g \) then from Theorem 2 it follows that two distinct solutions of equation \( \Delta^m y_n + f(n, y_n) = 0 \) diverge with velocity not smaller then \( Cn^{m-1} \) where \( C \) is a positive constant.

The theorem given below can be proved similarly as Theorem 1.

**Theorem 3.** Let \( y \) and \( z \) are solutions of the equations

(E3) \[ \Delta(r_n^{m-1} \Delta(\ldots \Delta(r_n \Delta y_n) \ldots )) + f(n, y_n) = 0, \quad n \in \mathbb{N}, \]

(E4) \[ \Delta(r_n^{m-1} \Delta(\ldots \Delta(r_n^1 \Delta z_n) \ldots )) + g(n, z_n) = 0, \quad n \in \mathbb{N}, \]

respectively, where \( r_n^i : N \rightarrow R_+ \), \( i = 1, 2, \ldots, m-1 \) are nondecreasing functions, \( f \), \( g : N \times R \rightarrow R_+ \). Let \( g(n, x) \geq f(n, x) \) for all \( n \geq 1 \) and \( x \in R \), \( f \) or \( g \) is a nondecreasing function with respect to the last argument. If \( y \) and \( z \) satisfy the conditions

(7) \[ \Delta(r_1^{\mu} \Delta(\ldots \Delta(r_1^{\mu} \Delta(y_1-z_1))) \geq 0, \quad \mu = 2, \ldots, m-2 \]

then

(8) \[ \Delta(r_{n+1}^{\mu} \Delta(\ldots \Delta(r_{n+1}^{\mu} \Delta(y_{n+1}-z_{n+1}))) \geq 0, \quad n \geq 1, \quad \mu = 2, \ldots, m-2. \]

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