ON THE INDEFINITE INTEGRAL
OF A STEPANOV’S ALMOST PERIODIC FUNCTION

In the paper the author proves that if the indefinite integral of an $S$-almost periodic function is bounded, then this integral is a $V$-almost periodic function.

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A set $E \subseteq (-\infty, \infty)$ is called relatively dense iff there is a positive number $l$ such that in each open interval $(\alpha, \alpha+l)$, $\alpha \in (-\infty, \infty)$, there is at least one element of the set $E$.

Let $x$ be a continuous function defined on the whole real axis and taking real values. If for $\varepsilon > 0$ there is

$$\sup_{-\infty < u < \infty} |x_{\tau}(u) - x(u)| \leq \varepsilon,$$

where $x_{\tau}(u) \equiv x(u + \tau)$, then the number $\tau$ is called $\varepsilon$ - almost period of $x$. Let us denote the set of $\varepsilon$ - almost periods of $x$ by $E\{\varepsilon;x\}$. The function $x$ is called uniformly almost periodic or a Bohr’s almost periodic function (B-a.p.) iff for each $\varepsilon > 0$ the set $E\{\varepsilon;x\}$ is relatively dense. For example the function $x$ of the form

$$(1) \quad x(u) = \sin u + \sin(\sqrt{2}u)\quad \text{for } u \in (-\infty, \infty)$$

is uniformly almost periodic and $x$ is not periodic. (See [1]).

Let us denote by $L^p$, where $p \geq 1$, the space of real functions $x$, measurable in the sense of Lebesgue for which

$$\int_a^b |x(s)|^p \, ds < \infty$$

for arbitrary $a, b \in (-\infty, \infty)$. For $x, y \in L^p$ let us put
\[ D_{sp}(x,y) = \sup_{-\infty < u < \infty} \left\{ \int_{u}^{u+1} |x(s) - y(s)|^p ds \right\}^{1/p} \]

If for \( x \in L^p \) and for \( \varepsilon > 0 \) there is \( D_{sp}(x \tau , x) \leq \varepsilon \), then the number \( \tau \) is called \( S^p, \varepsilon \)-almost period of \( x \). The function \( x \in L^p \) is called Stepanov's almost periodic function \((S^p - a.p.)\) iff for each \( \varepsilon > 0 \) the set \( E_{s,p}(\varepsilon, x) \) of \( S^p, \varepsilon \)-almost periods of \( x \) relatively dense. If \( p = 1 \) we have an \( S \)-a.p. function. For example the continuous function
\[
x(u) = \sin \frac{1}{2 + \cos u + \cos(\sqrt{2}u)}
\]
for \( u \in (-\infty, \infty) \) is \( S \)-a.p. and \( x \) not \( B \)-a.p. (See[1]).

Let \( X_0 \) be the set of functions defined on the whole real axis taking finite real values. Let us denote for an arbitrary \( t \in (-\infty, \infty) \) the Jordan variation of the function \( x \in X_0 \) on the interval \( <t - 1, t + 1> \) by \( V(x; t) \).

For \( x \in X_0 \) let us write
\[
V(x) = \sup_{-\infty < t < \infty} \{ |x(t) + V(x; t)| \}.
\]

We say that \( x \in X_0 \) satisfies the condition \((W)\) iff for every \( \alpha \in (-\infty, \infty) \) and for every \( l > 0 \) there exists \( M > 0 \) such that for every \( t \in (\alpha, \alpha + l) \) we have \( V(x; t) \leq M \). Let us put
\[ \tilde{X}_0 = \{ x \in X_0 : x \text{ is continuous and satisfies the condition } (W) \}. \]
The function \( x \in \tilde{X}_0 \) is called almost periodic in variation \((V \text{-a.p.)}\) iff for \( \varepsilon > 0 \) the set \( E_v(\varepsilon, x) \) of \( V \)-almost periods of \( x \), i.e. the set of numbers \( \tau \) for which \( V(x \tau - x) \leq \varepsilon \), is relatively dense. Every \( V \)-a.p. function is a Bohr's a.p. function. For example the Bohr's a.p. function \( x \) of the form (1) is \( V \)-a.p. Let us write \( x(u) = x_1(u) + x_2(u) \) for \( u \in (-\infty, \infty) \), where
\[
x_1(u) = \begin{cases} 0 & \text{for } u = k \\ (u - k) \sin \frac{\pi}{u - k} & \text{for } u \in (k, k + 1) \end{cases}, \quad k = 0, \pm 1, \pm 2, \ldots
\]
\( x_2(u) = \sin(\sqrt{2}u) \quad \text{for} \ u \in (-\infty, \infty). \)

Then \( x \) is a Bohr's a.p. function and \( x \) is not \( V \)-a.p. (See[2]).

In [3] it was shown that if the indefinite integral of an \( S \)-a.p. function bounded and uniformly continuous, then this integral is \( V \)-a.p. The following theorem is true:

**Theorem 1.** If \( x \) is an \( S \)-a.p. function and the indefinite integral of \( x \)

\[
F(u) = \int_{u_o}^{u} x(s)ds + C \quad \text{for} \ u \in (-\infty, \infty)
\]

is bounded, then \( F \) is \( V \)-a.p.

**Proof.** Let \( x \) be \( S \)-a.p. and \( S_x(\cdot;h) \) be the Steklov function of \( x \) of the form

\[
S_x(u;h) = \frac{1}{2h} \int_{u-h}^{u+h} x(s)ds,
\]

where \( h > 0, \ u \in (-\infty, \infty). \ S_x(\cdot;h) \) is \( B \)-a.p. Let us denote

\[
F_x(w;h) = \int_{w_o}^{w} S_x(u;h)du + C \quad \text{for} \ h > 0,
\]

where \( w \in (-\infty, \infty). \)

For \( w \in (-\infty, \infty) \) we have

\[
|F_x(w;h) - F(w)| \leq \frac{1}{2h} \int_{-h}^{h} \{ | \int_{w}^{w+s} x(u)du| + | \int_{w_o}^{w+s} x(u)du| \} dx.
\]

Let us choose an \( S, \varepsilon/2 \)-almost period \( \tau \) of \( x \) such that

\( \tau \in (-w, -w + l) \), where \( l = l(\varepsilon) > 0 \) is the number which characterizes the relative density of the set \( E_{s,1} \{ \varepsilon/2; x \} \). For \( 0 \leq s \leq 1 \) we obtain

\[
\int_{w}^{w+s} |x(u + \tau)|du \leq \int_{0}^{l+1} |x(u)|du < \infty,
\]
because $x$ is $S^1$-bounded. Hence there exists $\Delta = \Delta(\varepsilon) > 0$ such that for $0 \leq s < \Delta$ we have

$$\int_{w}^{w+s} |x(u+\tau)|du < \frac{\varepsilon}{2}$$

and

$$\left| \int_{w}^{w+s} x(u)du \right| < \int_{w}^{w+s} |x(u+\tau) - x(u)|du + \frac{\varepsilon}{2} \leq \varepsilon.$$

Therefore we obtain the following estimation

$$\left| \int_{w}^{w+s} x(u)du \right| < \varepsilon \quad \text{for } |s| < \Delta$$

uniformly with respect to $w \in (-\infty, \infty)$, and so the sequence $(F_x(w; h_n))$, where $h_n \to 0$, is convergent to $F(w)$ uniformly with respect to $w \in (-\infty, \infty)$. The Steklov function $S_x(\cdot; h)$ of $x$ satisfies the following inequality

$$|S_x(u; h) - S_x(u; h)| \leq \frac{1}{2h} \int_{u-h}^{u+h} |x(s+\tau) - x(s)|ds \quad \text{for every } u \in (-\infty, \infty).$$

Hence $S_x(\cdot; h)$ is $B$-a.p. Because for every $h > 0$ and every $w \in (-\infty, \infty)$

$$F_x(w; h) = \frac{1}{2h} \int_{-h}^{h} [F(w+s) - F(w_0 + s)]ds + C$$

and $F$ is bounded, $F_x(\cdot; h)$ is also bounded on $(-\infty, \infty)$. Therefore $F_x(\cdot; h)$ is $B$-a.p. The limit $F$ of the sequence $(F_x(\cdot; h_n))$, where $h_n \to 0$, which is uniformly convergent, is $B$-a.p.

Because $x$ is $S^1$-bounded, for every $t \in (-\infty, \infty)$ we have

$$V(F; t) \leq \int_{t-1}^{t+1} |x(s)|ds \leq M,$$

where $M$ is a constant, and so we see that $F \in \tilde{X}_0$. For an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for $n_0 > N$ we have
uniformly with respect to $w \in (-\infty, \infty)$. It is known (see [1], p.29) that there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that $\varepsilon' < \varepsilon/15$ and every $\varepsilon'$ - almost period of $S_x(\cdot; h_{n_0})$ is an $\varepsilon/3$-almost period of $F_x(\cdot; h_{n_0})$. Hence for $\tau \in E_{\delta_1} \{2h_{n_0}, \varepsilon', \varepsilon\}$, where $h_{n_0} \leq 1$, we obtain

$$V(F_\tau - F) \leq \sup_{-\infty < t < \infty} |F(t + \tau) - F_x(t + \tau; h_{n_0})| +$$

$$+ \sup_{-\infty < t < \infty} |F_x(t + \tau; h_{n_0}) - F_x(t; h_{n_0})| + \sup_{-\infty < t < \infty} |F_x(t; h_{n_0}) - F_x(t)| +$$

$$+ \sup_{-\infty < t < \infty} V(F_\tau - F; t) \leq \frac{7}{15} \varepsilon + \sup_{-\infty < t < \infty} \int_{t-1}^{t+1} |x(s + \tau) - x(s)| ds \leq \varepsilon,$$

and so $F$ is $V$-a.p.

**Theorem 2.** Let us assume that $x$ an $S$-a.p. function.

a) If the indefinite integral of $x$

$$F(u) = \int_{u_0}^{u} x(s) ds + C \quad \text{for} \quad u \in (-\infty, \infty)$$

satisfies the following condition

$$\sup_{-\infty < t < \infty} \left| \int_{t}^{t+1} F(u) du \right| < \infty,$$

then the function $G$ of the form

$$G(w) = \int_{w}^{w+1} F(u) du + C \quad \text{for} \quad w \in (-\infty, \infty)$$

is $B$-a.p.

b) If the indefinite integral of $x$ is $S^{1}$-bounded, then $G$ is $V$-a.p.

**Proof.** a) Let $x$ be $S$-a.p. and let $S_x(\cdot; h)$ be the Steklov function of $x$ of the form (2). Let us write for $w \in (-\infty, \infty)$
\[ F_x(w; h) = \int_{w_0}^{w} S_x(u, h) du + C. \]

Similarly as in the proof of Theorem 1 we obtain that the sequence \((F_x(w; h_n))\), where \(h_n \to 0\), is convergent to \(F(w)\) uniformly with respect to \(w \in (-\infty, \infty)\).

Because \(x\) is \(S\)-a.p., for every fixed \(s \in (-\infty, \infty)\) and every fixed \(t \in (-\infty, \infty)\) the function \(y_{st}\) of the form

\[ y_{st}(u) = \int_{s+t+u}^{s+t+u+1} x(w) dw \]

is \(B\)-a.p. By the assumption it follows that there exists a constant \(M > 0\) such that for every \(t \in (-\infty, \infty)\) we have

\[ \left| \int_{t}^{t+1} F(r) dr \right| \leq M. \]

Hence for every \(r \in (-\infty, \infty)\) we obtain

\[ \left| G_{st}(r) \right| = \left| \int_{r_0}^{r} y_{st}(u) du + C \right| \leq 2M + |C|, \]

and so \(G_{st}\) \(B\)-a.p. It is known (see [1], p.29) that for an arbitrary \(\varepsilon > 0\) there exists \(0 < \varepsilon' = \varepsilon'(\varepsilon) < \varepsilon\) such that every \(\varepsilon'\) - almost period of \(y_{st}\) is an \(\varepsilon\) - almost period of \(G_{st}\). Hence for \(\tau \in E_{S^1}\{\varepsilon', x\}\) we have \(\tau \in E\{\varepsilon; G_{st}\}\) and

\[ \left| \int_{0}^{\tau} y_{st}(u) du \right| \leq \varepsilon \]

for every \(s\) and every \(t\). Therefore for every \(t \in (-\infty, \infty)\) and for \(\tau \in E_{S^1}\{\varepsilon', x\}\) we obtain

\[ \left| \int_{t+\tau}^{t+1} F_x(w; h) dw - \int_{t}^{t+1} F_x(w; h) dw \right| \leq \frac{1}{2h} \int_{-h}^{h} \int_{0}^{\tau} y_{st}(u) du \, ds \leq \varepsilon, \]
i.e. $E_{S^1}^{t+1}\{\varepsilon'; x\} \subset E\{\varepsilon; z_h\}$, where

$$z_h(t) = \int_{t}^{t+1} F_x(w; h)dw,$$

and hence $z_h$ is $B$-a.p. The sequence $(z_{h_n}(t))$ is convergent to $G(t)$ for every $h_n \to 0$ uniformly with respect to $t \in (-\infty, \infty)$, and so $G$ is $B$-a.p.

b) Because $F$ is $S^1$-bounded, for every $t \in (-\infty, \infty)$ we have

$$V(G; t) \leq \int_{t-1}^{t+1} |F(u)|du + \int_{t}^{t+2} |F(u)|du \leq M,$$

where $M$ is a constant, and so $G \in \tilde{X}_o$.

For an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for $n_o > N$ we obtain

$$|z_{h_{n_0}}(t) - G(t)| \leq \frac{\varepsilon}{9},$$

uniformly with respect to $t \in (-\infty, \infty)$, where $z_{h_{n_0}}$ is of the form (3). It is known that there exists $0 < \varepsilon' = \varepsilon'(\varepsilon) < \varepsilon/9$ such that $E_{S^1}^{t+1}\{\varepsilon'; x\} \subset E\{\varepsilon/3; z_{h_{n_0}}\}$. Hence for every $t \in (-\infty, \infty)$ and every $\tau \in E_{S^1}^{t+1}\{\varepsilon'; x\}$ we have

$$|G(t + \tau) - G(t)| \leq \frac{5}{9} \varepsilon$$

and

$$V(G_t - G) \leq \sup_{-\infty < t < \infty} |G(t + \tau) - G(t)| + 4 \sup_{-\infty < t < \infty} \int_{t}^{t+1} |x(s + \tau) - x(s)|ds < \varepsilon,$$

i.e. $G$ is $V$-a.p.
References


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