A NOTE ON \( m \)-PARABOLIC PROBLEM
FOR SPHERICAL SHAPE

The subject of the paper is the problem of the existence and of the uniqueness to
the polyharmonic problem for spherical shape with the limit conditions of Riquier type.
In [4] this problem by another method is solved. By the suitable reduction of the initial
conditions the construction of the solution is reduced to the construction given in [7].
By the formulas of [7] the explicit form of the solution is obtained.
The uniqueness theorem is also given, which is not given in [7].
Key words: polyharmonic problem, initial-boundary conditions, uniqueness and
existence of the solution, Green function, Green potentials.

1. Introduction

The subject of the paper is the existence and uniqueness of the radial
solution
\[
U(r, t) = u(x, t), \quad r = |x|,
\]
\[
P^m u(x, t) = f(x, t), \quad x = (x_1, x_2, x_3),
\]
\[
P = \Delta - D_r, \quad \Delta = \sum_{i=1}^{3} D_{x_i}^2, \quad P^m = P(P^{m-1})
\]
\[
m \in N - \{1\}, \quad P^1 = P,
\]
in spherical shape
\[
D = \left\{ (x, t) : a < |x| < b, \quad 0 < a < b, \quad t \in (0, T), \quad |x| = \left( \sum_{i=1}^{3} x_i^2 \right)^{1/2} \right\}.
\]
The radial solution \( u(x, t) = U(r, t), \quad r = |x| \) satisfies the initial conditions
\[
P^{j-1}u(x, 0) = f_j(x) \quad \text{for} \quad (x, t) \in D_1 = \{(x, 0) : a < |x| < b\}, \quad j = 1, 2, \ldots, m,
\]
and the boundary conditions
\[
P^{j-1}u(x, t) = h_j(t) \quad \text{for} \quad (x, t) \in S_1 = \{(x, t) : |x| = a, t \in (0, T)\}, \quad j = 1, 2, \ldots, m,
\]
(4) \( P^{j-1} u(x, t) = k_j(t) \) for \((x, t) \in S_z = \{(x, t) : |x| = b, \ t \in (0, T]\}, \)
\[ j = 1, 2, \ldots, m. \]

In [4] the radial solution to the problem (1) – (4) by another method is given.

In [7] the similar limit problem with another initial conditions is examined but without uniqueness.

In the papers [1] and [2] the poly-parabolic problem for the strip and for the quart-time-plane with boundary conditions of Lauricella type are considered respectively.

In [3] the poly-parabolic problem for the spatial three-dimensional cylinder with boundary conditions of Riquier type is solved.

In the paper [6] the \( m \)-parabolic problem for the strip with boundary conditions of even type is examined.

2. The radial problem

Let
\[
V(r, t) = r U(r, t) \quad \text{and} \quad Q = D_r^2 - D_t.
\]
By (5) and by [7], p.118, we obtain the following

Lemma 1. If \( u \in C^{2m, m}(D), \ u(x, t) = U(r, t), \ r = |x| \) then
1° \( U \in C^{2m, m}((a, b) \times (0, T)) \).
2° \( P^m u(x, t) = r^{-1} Q^m V(r, t) = r^{-1} Q^m V(r, t) \).

By Lemma 1 and by (1) – (4) problem we obtain

Lemma 2. If the function \( U \in C^{2m, m}((a, b) \times (0, T]) \) is the solution of the (1) – (4) problem, then the function \( V(r, t) \in C^{2m, m}(D_2), \ D_2 = \{(r, t) : a < r < b, \ t \in (0, T)\} \) and satisfies the conditions

(1a) \( Q^m V(r, t) = F(r, t), \ F(r, t) = r f(r, t), \ (r, t) \in D_2, \)
(2a) \( Q^{j-1} V(r, 0) = F_j(r), \ r \in (a, b), \ F_j(r) = r f_j(r), \ j = 1, 2, \ldots, m, \)
(3a) \( Q^{j-1} V(a, t) = H_j(t), \ t \in (0, T], \ H_j(t) = a h_j(t), \ j = 1, 2, \ldots, m, \)
(4a) \( Q^{j-1} V(b, t) = K_j(t), \ t \in (0, T], \ K_j(t) = b k_j(t), \ j = 1, 2, \ldots, m. \)
Conversely. If $V \in C^{2m,m}(D_2)$ and satisfies (1a) – (4a), then the function $u(x,t) = U(r,t)|_{r=\mid x\mid} = |x|^m V(r,t)$ is the solution of the (1) – (4) problem.

Definition 1. Denote by $(K_1)$ the class of all functions $F : D_2 \to R$ such that $F \in C^{1,0}(D_2) \cap C^{0,0}(\overline{D}_2)$.

Definition 2. Denote by $(K_2)$ the class of all functions $E : [a,b] \to R$, such that $E \in C^{4m-2}([a,b])$ and $D_{r}^{i-1} E(a) = D_{r}^{i-1} E(b) = 0$, $i = 1,2,\ldots,m$, $m \in N$.

Definition 3. Denote by $(K_3)$ the class of all functions $H : [0,T] \to R$, such that $H \in C^{1}([0,T])$ and $D_{t}^{i-1} H(0) = 0$, $i = 1,2,\ldots,m$, $m \in N$.

Definition 4. Denote by $(K_4)$ the class of all functions $V : D \to R$, such that $V \in C^{2m,m}(D_2) \cap C^{m, m-1}(\overline{D}_2)$.

3. Uniqueness theorem

Let $W(r,t) = w_1(r,t) - w_2(r,t)$ and

$D(t) = \{(r,s) : a < r < b, \ 0 < s < t, \ t < T\}$.

Theorem 1. If $w_1(r,t), w_2(r,t) \in (K_4)$ are solutions of the problem (1a) – (4a), then $w_1(r,t) = w_2(r,t)$ for $(r,t) \in \overline{D}_2$.

Proof. For the function $W(r,s)$ we have

$$Q^m W(r,s) = 0,$$

$$Q^{i-1} W(r,0) = 0, \ a < r < b, \ j = 1,2,\ldots,m,$$

$$Q^{i-1} W(a,s) = 0, \ 0 < s < t, \ t < T, \ j = 1,2,\ldots,m,$$

$$Q^{i-1} W(b,s) = 0, \ 0 < s < t, \ t < T, \ j = 1,2,\ldots,m.$$

Multiplying by $Q^{m-1} W(r,s)$ both sides of the equation (6) and integrating over $D(t)$, we obtain

$$\int_{0}^{b} \int_{a}^{b} Q^{m-1} W(r,s) Q^{m} W(r,s) dr ds = I_1 + I_2 = 0,$$
where
\[
I_1 = - \int_0^t \int_0^b Q^{m-1} W(r,s) D_r Q^{m-1} W(r,s) dr \, ds,
\]
\[
I_2 = \int_0^t \int_0^b Q^{m-1} W(r,s) D_r^2 Q^{m-1} W(r,s) dr \, ds.
\]
Integrating \( I_1 \) by parts we obtain
\[
I_1 = - \int_0^b (Q^{m-1} W(r,s))^2 \left|_{s=t}^{s=o} \right. dr + \int_0^b \int_0^t Q^{m-1} W(r,s) D_r Q^{m-1} W(r,s) dr \, ds =
\]
\[
= - \int_0^b (Q^{m-1} W(r,t))^2 dr + \int_0^b (Q^{m-1} W(r,0))^2 dr.
\]
Consequently by (7) we have
\[
(11) \quad I_1 = - \frac{1}{2} \int_a^b (Q^{m-1} W(r,t))^2 dr.
\]
Integrating \( I_2 \) by parts we obtain
\[
I_2 = \int_0^t (Q^{m-1} W(r,s) D_r Q^{m-1} W(r,s)) \bigg|_{r=a}^{r=b} ds - \int_0^b \int_a^t (D_r Q^{m-1} W(r,s))^2 dr \, ds.
\]
Consequently by (8) and (9) we have
\[
(12) \quad I_2 = - \int_0^t \left( \int_a^b D_r Q^{m-1} W(r,s) \right)^2 dr \, ds \leq 0.
\]
By (10), (11) and (12) we obtain
\[
I_1 + I_2 = 0
\]
and
\[
Q^{m-1} W(r,t) = 0, \quad (r,t) \in D_2.
\]
Similarly we obtain
\[
Q^i W(r,t) = 0, \quad (r,t) \in D_2, \quad i = 1, 2, \ldots, m-2.
\]
For \( i = 1 \) we have.
(13) \( QW(r, t) = 0, \quad (r, t) \in D_2, \quad \text{and} \quad QW(r, s) = 0, \quad (r, s) \in D(t). \)

Multiplying (13) by \( W(r, s) \) we obtain

(14) \( W(r, s)QW(r, s) = 0, \quad (r, s) \in D(t), \)

Integrating (14) over \( D(t) \), we obtain

\[
W(r, t) = 0, \quad (r, t) \in D_2, \quad \text{i.e.} \quad w_1(r, t) = w_2(r, t) \quad \text{for} \quad (r, t) \in D_2.
\]

By the continuity of \( W(r, t) \) in \( D_2 \), we obtain

\[
w_1(r, t) = w_2(r, t) \quad \text{for} \quad (r, t) \in \overline{D_2}.
\]

4. Reduction of the problem (1) – (4) to the problem with homogeneous initial conditions

Let

\[
w(r, t) = V(r, t) - R(r, t),
\]

where

\[
R(r, t) = f_o(r) + \sum_{j=1}^{m-1} (t^j/j!) \sum_{k=0}^{j} \binom{j}{k} \Delta^{j-k} f_k(r).
\]

**Lemma 3.** If \( F \in (K_1), \ F_j \in (K_2), \ j = 0, 1, \ldots, m-1, \ H_j, K_j \in (K_3), \ j = 0, 1, \ldots, m-1 \) and \( V \in (K_4) \) is solution of the (1a) – (4a) problem, then \( w \in (K_4) \) and satisfies the conditions:

(1b) \( Q^m w(r, t) = L(r, t), \quad L(r, t) = F(r, t) - Q^m R(r, t), \quad (r, t) \in D_2, \)

(2b) \( Q^{j-1} w(r, 0) = 0, \quad j = 1, 2, \ldots, m, \quad a < r < b, \)

(3b) \( Q^{j-1} w(a, t) = \overline{H}_j(t), \quad \overline{H}_j(t) = H_j(t) - Q^{j-1} R(a, t), \quad j = 1, 2, \ldots, m, \quad \overline{0} < t < T, \)

(4a) \( Q^{j-1} w(b, t) = \overline{K}_j(t), \quad \overline{K}_j(t) = K_j(t) - Q^{j-1} R(b, t), \quad j = 1, 2, \ldots, m, \quad 0 < t < T \)

Conversely. If \( L \in (K_1), \ \overline{H}_j, \overline{K}_j \in (K_3), \ j = 0, 1, \ldots, m-1 \) and the function \( w \in (K_4) \) satisfies (1b) – (4b), then the function

\[
V(r, t) = w(r, t) + R(r, t) \in (K_4) \quad \text{and} \quad \text{satisfies (1a) – (4a)}.
\]

We omit the simple proof.
5. Green function

By [5], p.476, it is known that the Green function for the equation $QV = 0$ for the strip and Dirichlet boundary conditions is of the form:

$$ G(r,t;p,s) = U_0 + H(r,t;p,s) $$

where

$$ U_0(r,t;p,s) = (t-s)^{-1/2} \exp(B(t,s)(r_0^j - p)^2), $$

$$ B(t,s) = (-4(t-s))^{-1}, \quad r_0^i = r, \quad i = 1,2. $$

$$ H(r,t;p,s) = -U_1^1(r,t;p,s) + \sum_{n=0}^{\infty} (-U_1^{12}_{2n+3}(r,t;p,s) + U_2^{22}_{2n+2}(r,t;p,s)) + $$

$$ + U_2^{22}_{2n+2}(r,t;p,s) = \sum_{n=0}^{\infty} (U_2^{22}_{2n+3}(r,t;p,s) + U_2^{22}_{2n+2}(r,t;p,s)) + $$

or

$$ H(r,t;p,s) = -U_1^2(r,t;p,s) + \sum_{n=0}^{\infty} (U_2^{22}_{2n+3}(r,t;p,s) + U_2^{22}_{2n+2}(r,t;p,s)) + $$

$$ + \sum_{n=0}^{\infty} (U_2^{22}_{2n+2}(r,t;p,s) - U_1^{11}_{2n+1}(r,t;p,s)). $$

$$ U_n^j(r,t;p,s) = (t-s)^{-1/2} \exp(B(t,s)(r_n^j - p)^2), \quad j = 1,2, \quad n = 0,1,2,... $$

and

$$ r_0^1 = r_0^2 = r $$

$$ r_2^1 = r + 2n(b-a) $$

$$ r_2^{n+1} = -r - 2n(b-a) + 2a $$

$$ r_2^2 = r - 2n(b-a) $$

$$ r_2^{n+1} = -r + 2n(b-a) + 2b $$

Let us consider the function $G_1$ given by formulas:

$$ G(r,t;p,s) = (i!) (t-s)^i G(r,t;p,s), \quad i = 1,2,...,m-1. $$

By [7], p.120, we have.

**Lemma 4.** When $0 < s < t$, then the functions $G_i$ satisfy the following conditions:

1° $Q^i G_{i-1}(r,t;p,s) = 0, \quad i = 2,3,...,m, \quad (r,t),(p,s) \in D_1 ,$

2° $Q^i G_{i+1}(a,t;p,s) = Q^i G_{i+1}(b,t;p,s) = 0, \quad i = 0,1,2,...,m-1, \quad (r,t),(p,s) \in D_1.$
By [7], p.121, we have.

**Lemma 5.** If \( 0 < s < t < T \), then

\[
D_p G(r, t; p, s) \big|_{p=a} = S_1(r, t, s) + S^1_1(r, t, s),
\]

where

\[
S_1(r, t, s) = D_p \left( U_0(r, t; p, s) - U^{1}_1(r, t; p, s) \right) \big|_{p=a} = -2(t - s)^{-1/2} B(t, s)(r - a) \exp(B(t, s)(r - a)^2),
\]

\[
S^1_1(r, t, s) = -4(t - s)^{-1/2} B(t, s) \sum_{n=0}^{\infty} A_n \exp(B(t, s)(A_n)^2) =
\]

\[
= -4(t - s)^{-1/2} B(t, s) \sum_{n=0}^{\infty} B_n \exp(B(t, s)(B_n)^2)
\]

and \( A_n = -r - 2(n+1)(b - a) + a, \quad B_n = r - 2(n+1)(b - a) - a, \quad D_p G(r, t; p, s) \big|_{p=b} = S_2(r, t, s) + S^1_2(r, t, s); \)

where

\[
S_2(r, t, s) = D_p \left( U_0(r, t; p, s) - U^{2}_1(r, t; p, s) \right) \big|_{p=b} = -2A(t - s)B(t, s)(r - b) \exp(B(t, s)(r - b)^2),
\]

\[
S^1_2(r, t, s) =
\]

\[
= -4(t - s)^{-1/2} B(t, s) \sum_{n=0}^{\infty} \left[ D_n \exp(B(t, s)(D_n)^2) C_n \exp(B(t, s)C_n^2) \right],
\]

\[
C_n = r + 2(n+1)(b - a) - 2a + b, \quad D_n = r - 2(n+1)(b - a) - b.
\]

6. **Green iterated potentials**

Similarly to the [7], p.123, let us consider the following potentials

\[
w^1_0(r, t) = A_1 \int_0^t D_p G(r, t; p, s) \big|_{p=a} \overline{H}_0(s) \, ds = T^{11}_0(r, t) + T^{12}_0(r, t),
\]

\[
w^2_0(r, t) = A_1 \int_0^t D_p G(r, t; p, s) \big|_{p=b} \overline{K}_0(s) \, ds = T^{21}_0(r, t) + T^{22}_0(r, t),
\]

where

\[
T^{11}_0(r, t) = A_1 \int_0^t S_1(r, t, s) \overline{H}_0(s) \, ds,
\]

\[
T^{12}_0(r, t) = A_1 \int_0^t S^1_1(r, t, s) \overline{H}_0(s) \, ds,
\]

\[
T^{21}_0(r, t) = A_1 \int_0^t S_2(r, t, s) \overline{K}_0(s) \, ds,
\]

\[
T^{22}_0(r, t) = A_1 \int_0^t S^1_2(r, t, s) \overline{K}_0(s) \, ds.
\]
\[ T_0^{21}(r,t) = A_1 \int_0^t S_2(r,t,s) K_0(s) \, ds , \]
\[ T_0^{22}(r,t) = A_1 \int_0^t S_2'(r,t,s) K_0(s) \, ds . \]

Let
\[ w_1^1(r,t) = A_2 \int_0^t \int_a^b G(r,t; p, s) M_1(p, s) \, dp \, ds , \]
where
\[ M_1(p, s) = M_1^1(p, s) + M_1^2(p, s) , \]
and
\[ M_1^1(p, s) = \int_0^s S_1(p, s, s_1) H_1(s_1) \, ds_1 , \]
\[ M_1^2(p, s) = \int_0^s S_2^1(p, s, s_1) H_1(s_1) \, ds_1 . \]

and
\[ w_1^2(r,t) = A_2 \int_0^t \int_a^b G(r,t; p, s) N_1(p, s) \, dp \, ds , \]
where
\[ N_1(p, s) = N_1^1(p, s) + N_1^2(p, s) \]
and
\[ N_1^1(p, s) = \int_0^s S_2(p, s, s_1) K_1(s_1) \, ds_1 , \]
\[ N_1^2(p, s) = \int_0^s S_2^1(p, s, s_1) K_1(s_1) \, ds_1 . \]

Let
\[ w_2^1(r,t) = A_3 \int_0^t \int_a^b G_1(r,t; p, s) M_2(p, s) \, dp \, ds , \]
where
\[ M_2(p, s) = \int_0^s D_{p_1} G(p, s; p_1, s_1) \bigg|_{p_1=b} H_1(s_1) \, ds_1 , \]
and so one ................................................................................................................
\[ w_{m-1}^1(r, t) = A_{m-1} \int_{0}^{b} \int_{a}^{b} G_{m-2}(r, t; p, s) M_{m-1}(p, s) \, dp \, ds, \]

where

\[ M_{m-1}(p, s) = \int_{0}^{s} D_{p, 1} G(p, s, p_1, s_1) \bigg|_{p_1 = a} \bar{H}_{m-1}(s_1) \, ds_1, \]

\[ w_{m-1}^2(r, t) = A_{m-1} \int_{b}^{t} \int_{a}^{b} G_{m-2}(r, t; p, s) N_{m-1}(p, s) \, dp \, ds, \]

where

\[ N_{m-1}(p, s) = \int_{0}^{s} D_{p, 1} G(p, s, p_1, s_1) \bigg|_{p_1 = b} \bar{K}_{m-1}(s_1) \, ds_1, \]

and

\[ w_m(r, t) = A_{m-1} \int_{0}^{t} \int_{a}^{b} G_{m-1}(r, t; p, s) L(p, s) \, dp \, ds, \]

\[ A_i \left[ 2\sqrt{\pi} \right]^{\frac{1}{2}} (i!)^{-1}, \quad i = 1, 2, \ldots, m-1. \]

Let

\[ w_m(r, t) = \sum_{i=1}^{4} w_{m}^i(r, t), \]

where

\[ w_m^1(r, t) = A_{m-1} \int_{0}^{t} \int_{0}^{b} (t-s)^{m-1} U_0(r, t; p, s) L(p, s) \, dp \, ds = \]

\[ = A_{m-1} \int_{0}^{t} \int_{a}^{b} \exp(B(t, s)(r-p))^2 L(p, s) \, dp \, ds, \]

\[ w_m^2(r, t) = A_{m-1} \int_{0}^{t} \int_{0}^{b} (t-s)^{m-1} U_1^1(r, t; p, s) L(p, s) \, dp \, ds = \]

\[ = A_{m-1} \int_{0}^{t} \int_{a}^{b} \exp(B(t, s)(-r + 2a - p))^2 L(p, s) \, dp \, ds, \]

\[ w_m^3(r, t) = A_{m-1} \int_{0}^{t} \int_{0}^{b} (t-s)^{m-1} U_1^2(r, t; p, s) L(p, s) \, dp \, ds = \]
\[ = A_{m-1} \int_a^b \int_0^t (t,s)^{m-1} \exp(B(t,s)(r + 2b - p))^2 L(p,s) \, dp \, ds \, , \]

\[ w_m \cdot (r,t) = A_{m-1} \int_a^b \int_0^t (t-s)^{m-1} H^1(r,t,p,s)L_1(p,s) \, dp \, ds \, , \]

where

\[ H^1(r,t,p,s) = H(r,t,p,s) + U^1_1(r,t,p,s) + U^2_1(r,t,p,s) , \]

By [7], p.128, we have

**Lemma 6.** If the function \( L \) belong to the class of all functions \( L(p,s) \) such that the functions \( D^i D^j_p D^j_s L(p,s) \), \( 0 \leq i \leq m-1 \), \( 0 \leq j \leq m-1 \), are continuous and bounded in \( \bar{D} \) and

\[ D^i_s L(a,s) = D^i_s L(b,s) = 0 \, , \quad D^i_s L(p,0) = 0 \, , \quad i = 0,1,...,m-1 \]

then

1° \( Q \cdot w_m \cdot (r,t) = L(r,t) \) for \( (r,t) \in D_1 \),

2° \( Q^i w_m \cdot (r,0) = 0 \, , \quad i = 0,1,...,m-1 \) for \( r \in (a,b) \),

3° \( Q^i w_m \cdot (a,t) = 0 \, , \quad i = 0,1,...,m-1 \, , \quad t \in (0,T) \),

4° \( Q^i w_m \cdot (b,t) = 0 \, , \quad i = 0,1,...,m-1 \, , \quad t \in (0,T) \).

By [7], p.130, we have

**Lemma 7.** If the function \( \bar{H}_i, \bar{K}_i \), \( i = 0,1,...,m-1 \), belong to the class of all functions such that \( \bar{H}_i, \bar{K}_i \in C^4([0,T]) \) and 

\[ D^i_s \bar{H}_i(t) = D^i_s \bar{K}_i(t) = 0 \, , \quad i = 0,1,...,m-1 \]

then

1° \( Q^m w^i_j(r,t) = 0 \, , \quad j = 0,1,...,m-1 \, , \quad i = 1,2 \) for \( (r,t) \in D_1 \),

2° \( Q^k w^i_j(r,t) = 0 \, , \quad i = 1,2 \, , \quad k, j = 0,1,...,m-1 \),

3° \( Q^j w^i_j(r,t) \rightarrow 0 \) as \( r \rightarrow a \), if \( i \neq j \), \( Q^j w^i_j(r,t) \rightarrow \bar{H}_i(t) \) as \( r \rightarrow a \), \( i = 0,1,2,...,m-1 \), \( j = 0,1,...,m-1 \, , \quad t \in (0,T) \),

4° \( Q^j w^2_j(r,t) \rightarrow 0 \) as \( r \rightarrow b \), if \( i \neq j \), \( Q^j w^2_j(r,t) \rightarrow \bar{K}_i(t) \) as \( r \rightarrow b \), \( i = 0,1,...,m-1 \, , \quad j = 0,1,...,m-1 \, , \quad t \in (0,T) \).
5. Theorem on the existence of the solution

Theorem 2. If the assumptions of Lemmas 1 – 7 are satisfied, then the function

\[ w(r,t) = w_m(r,t) + \sum_{j=0}^{m-1} (w_j^1(r,t) + w_j^2(r,t)) \]

is the solution of the problem (1b) – (4b) and the function

\[ u(x,t) = U(r,t) \bigg|_{r=|x|} = |x|^{-1} \left( w(|x|,t) + r(|x|,t) \right) = \]

\[ = |x|^{-1} \left\{ w_m(|x|,t) + \sum_{j=0}^{m-1} (w_j^1(|x|,t) + w_j^2(|x|,t) + r(|x|,t) \right\} \]

is the solution of the problem (1) – (4).

References


(Institute of Mathematics, Kraków University of Technology, Warszawska 24, 31-155 Kraków). Received on 1.09.1993 and, in revised from, on 29.03.1995.