OSCILLATIONS OF SOME DIFFERENCE EQUATIONS

ABSTRACT. Oscillation criteria for nonlinear difference equation of the form
\[ \Delta(r_n \Delta(u_n + p_n u_{n-k})) + q_n f(\eta_{n-1}) = 0, \quad n = 0, 1, 2, \ldots, \]
are established. These criteria extend those in [13-15] and [18].

KEYWORDS: oscillations, nonoscillatory solution, difference equations.

1. INTRODUCTION

In this note we consider the nonlinear difference equation of the form
\[ \Delta(r_n \Delta(u_n + p_n u_{n-k})) + q_n f(\eta_{n-1}) = 0, \quad n = 0, 1, 2, \ldots, \]  \( (1) \)
where \( \Delta \) denotes the forward difference operator: \( \Delta v_n = v_{n+1} - v_n \) for any sequence \( (v_n) \) of real numbers, \( k \) and \( l \) are nonnegative integers, \( (p_n) \) and \( (q_n) \) are sequences of real numbers, \( (r_n) \) is a sequence of positive numbers and
\[ \sum_{m=0}^{\infty} \frac{1}{r_n} = \infty. \]  \( (2) \)
The function \( f \) is a real valued function satisfying \( uf(u) > 0 \) for \( u \neq 0 \). By a solution of \( (1) \) we mean a sequence \( (u_n) \) defined for \( n \geq -\max\{k, l\} \), which satisfies \( (1) \) for all large \( n \). A nontrivial solution \( (u_n) \) of \( (1) \) is said to be oscillatory if for every positive integer \( N \) there exists \( n \geq N \) such that \( u_n u_{n+1} \leq 0 \). Otherwise it is called nonoscillatory.

Recently, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions of „delay” and „neutral delay” type difference equations; see for example [2, 3, 6, 8-20] and Chapter 7 in the recent book by Győri and Ladas [4]. For the general theory of difference equations one can refer to [1, 5, 7].

Our purpose in this paper is to give sufficient conditions for the oscillation of solutions of \( (1) \). These criteria extend some results contained in [13-15] and [18].

2. MAIN RESULTS

The following theorem provides sufficient conditions for the oscillation of all solutions of \( (1) \).
Theorem 1. Assume that \( 0 \leq p_n \leq 1, q_n \geq 0 \) for \( n \geq n_0 \) and

\[
(3) \quad \frac{f(u)}{u} \geq \gamma > 0 \quad \text{for} \quad u \neq 0.
\]

If there exists a sequence \((h_n)\) such that \( h_n > 0 \) for \( n \geq n_0 \) and

\[
(4) \quad \sum_{n=n_0+1}^{\infty} \left[ \gamma h_n q_n (1 - p_{n-1}) - \frac{r_{n-1}(\Delta h_n)^2}{4h_n} \right] = \infty,
\]

then every solution of (1) is oscillatory.

Proof. Assume for the sake of contradiction that (1) has a nonoscillatory solution \((u_n)\), which we may assume (and we do) to be eventually positive. Set

\[
(5) \quad z_n = u_n + p_n u_{n-k}.
\]

By assumptions, there exists \( n_1 \geq n_0 \) such that \( z_n > 0 \) for \( n \geq n_1 \) and from (1) it follows that \( \Delta(r_n \Delta z_n) \leq 0 \) for \( n \geq n_1 \). Therefore \((r_n \Delta z_n)\) is a nonincreasing sequence. We claim that

\[
(6) \quad \Delta z_n > 0 \quad \text{for} \quad n \geq n_1.
\]

In fact, if there is an \( n_2 \geq n_1 \) such that \( \Delta z_{n_2} \leq 0 \), then there is an \( n_3 \geq n_2 \) that \( r_{n_3} \Delta z_{n_3} = c < 0 \) or \( r_{n_3} \Delta z_{n_3} = 0 \) for all \( n \geq n_2 \). In the first case we have \( r_n \Delta z_n \leq c \) for \( n \geq n_3 \) and so, bo (2), we get

\[
z_n \leq z_{n_3} + c \sum_{i=n_3}^{n-1} \frac{1}{r_i} \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty,
\]

which contradicts the fact that \( z_n > 0 \) for \( n \geq n_1 \).

In the second case, from (1) we would have \( q_n = 0 \) eventually, which is impossible in view of (4).

Next, observe that from (1) and (5), by (3), we have

\[
(7) \quad \Delta(r_n \Delta z_n) + \gamma q_n (z_{n-1} - p_{n-1} u_{n-1-k}) \leq 0.
\]

Since \( z_n \geq u_n \), (6) yields

\[
\Delta(r_n \Delta z_n) + \gamma q_n (z_{n-1} - p_{n-1} z_{n-1-k}) \leq 0,
\]

which in view of (6) we get eventually, say for \( n \geq n_4 \)

\[
(8) \quad \Delta(r_n \Delta z_n) + \gamma q_n (1 - p_{n-1}) z_{n-1} \leq 0.
\]

Define

\[
w_n = h_n \frac{r_n \Delta z_n}{z_{n-1}} \quad \text{for} \quad n \geq n_4.
\]

Therefore

\[
\Delta w_n = \frac{h_n \Delta(r_n \Delta z_n)}{z_{n-1}} + \frac{r_{n+1} \Delta z_{n+1} \Delta h_n}{z_{n+1}} - \frac{r_{n+1} \Delta z_{n+1} h_n (z_{n+1} - z_{n-1})}{z_{n-1} z_{n+1} z_{n-1}}.
\]
By (8) and (6), we get

\[ \Delta w_n \leq -\gamma h_n q_n (1 - p_{n-1}) + \frac{r_{n+1} \Delta z_{n+1} \Delta h_n - r_{n+1} \Delta z_{n+1} h_n \Delta z_{n-1}}{z_{n+1-1}^2}. \]

Using the fact that the sequence \((r_n \Delta z_n)\) is nonincreasing we have

\[ \Delta w_n \leq -\gamma h_n q_n (1 - p_{n-1}) + \frac{r_{n+1} \Delta z_{n+1} \Delta h_n}{z_{n+1-1}^2} - \frac{r_{n+1}^2 (\Delta h_n)^2 h_n}{r_{n-1} z_{n+1-1}^2} \]

and so

\[ \Delta w_n \leq -\gamma h_n q_n (1 - p_{n-1}) + \frac{r_{n-1} (\Delta h_n)^2}{4 h_n} - \left[ r_{n+1} \left( \frac{h_n}{r_{n-1}} \right)^{1/2} \frac{\Delta z_{n+1}}{z_{n+1-1}} - \frac{\Delta h_n}{2} \left( \frac{r_{n-1}}{h_n} \right)^{1/2} \right]^2 \leq -\gamma h_n q_n (1 - p_{n-1}) + \frac{r_{n-1} (\Delta h_n)^2}{4 h_n}, \quad n \geq n_5 = n_4 + l. \]

Summing both sides of the above inequality from \(n_5\) to \(n\) we obtain

\[ w_{n+1} \leq w_{n_5} - \sum_{i=n_5}^{n} \left[ \gamma h_i q_i (1 - p_{i-1}) - \frac{r_{i-1} (\Delta h_i)^2}{4 h_i} \right], \]

which in view of (4) leads to a contradiction as \(n \to \infty\). Thus the proof is complete.

**Remark 1.** If \(p_n = 0\) and \(f(u) = u\), then Theorem 1 and Theorem 2 of [14] are the same, and when \(r_n = 1\), \(h_n = 1\) and \(f(u) = u\), the Theorem 1 gives Corollary contained in [15].

**Theorem 2.** Let \(-1 \leq p_n \leq 0\), \(q_n \geq 0\) for \(n \geq n_0\) and \(f\) is a nondecreasing continuous function such that

\[ \int_{-\varepsilon}^{\varepsilon} \frac{du}{f(u)} < \infty, \quad \int_{-\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \quad \varepsilon > 0. \]

If

\[ \sum_{n=n_0}^{\infty} \frac{1}{p_n} \sum_{i=n+1}^{\infty} q_i = \infty, \]

then every unbounded solution of (1) is oscillatory.
Proof. Suppose that (1) has an unbounded nonoscillatory solution \( (u_n) \) and let it is eventually positive. Then for \( (z_n) \) defined in (5) we see as before that 
\[ \Delta(r_n \Delta z_n) \leq 0 \text{ eventually, that is } (r_n \Delta z_n) \text{ is nonincreasing. This implies that } (z_n) \text{ is eventually monotonic. Now, if } (z_n) \text{ is eventually nonpositive, then, by assumption, we have } u_n \leq -p_n u_{n-k} \leq u_{n-k}, \text{ which contradicts the assumption that } (u_n) \text{ is unbounded. Therefore } z_n > 0 \text{ eventually, say for } n \geq n_1 \geq n_0. \text{ Thus as in the proof of Theorem 1 we show that (6) holds. Since } 0 < z_n \leq u_n \text{ and } f \text{ is nondecreasing, we have}
\]
\[ \Delta(r_n \Delta z_n) + q_n f(z_{n-1}) \leq 0 \quad \text{for } n \geq n_2 = n_1 + l.
\]
Summing the above inequality from \( n \) to \( m \geq n \geq n_2 \) we get
\[ r_{m+1} \Delta z_{m+1} - r_n \Delta z_n + \sum_{i=n}^{m} q_i f(z_{i-1}) \leq 0.
\]
After letting \( m \to \infty \), we have
\[ \sum_{i=n}^{\infty} q_i f(z_{i-1}) \leq r_n \Delta z_n
\]
and so
\[ \sum_{i=n+l+1}^{\infty} q_i f(z_{i-1}) \leq r_n \Delta z_n.
\]
In view of monotonicity of \( (z_n) \) and \( f \) we see that
\[ \frac{1}{r_n} \sum_{i=n+l+1}^{\infty} q_i \leq \Delta z_n \leq \int_{z_n}^{z_{n+1}} \frac{du}{f(u)}, \quad \text{ for } n \geq n_2.
\]
Summing the last inequality from \( n_2 \) to \( n \), we obtain
\[ \sum_{j=n_2}^{n} \frac{1}{r_j} \sum_{i=j+i+1}^{\infty} q_i \leq \int_{z_{n_2}}^{z_{n+1}} \frac{du}{f(u)} < \int_{z_{n_2}}^{\infty} \frac{du}{f(u)} < \infty,
\]
which contradicts (9). The proof is similar when \( (u_n) \) is eventually negative.

Remark 2. In the case when \( r_n = 1 \), Theorem 2 reduces to Theorem 3 of [15].

The following criterion provides sufficient conditions for the oscillation of the difference of every solution of (1) when coefficient \( (q_n) \) is allowed to oscillate.

**Theorem 3.** If \( p_n = p \geq 0, f \) is a nondecreasing function and
\[ \sum_{n=0}^{\infty} q_n = \infty,
\]
then the difference \( (\Delta u_n) \) of every solution \( (u_n) \) of (1) oscillates.
Proof. If not, then (1) has a solution \((u_n)\) such that its difference \((\Delta u_n)\) is nonoscillatory. Assume that the sequence \((\Delta u_n)\) is eventually positive, say for \(n \geq n_0\). Thus \((u_n)\) is increasing for \(n \geq n_0\), which implies that \((u_n)\) is also nonoscillatory.

Denote
\[
z_n = u_n + p u_{n-k}
\]
and
\[
w_n = \frac{r_n \Delta z_n}{f(u_{n-1})}, \quad n \geq n_1 = n_0 + \max\{k, l\}.
\]
(11)

Then, by assumptions we have for \(n \geq n_1\)
\[
\Delta w_n = \frac{\Delta (r_n \Delta z_n)}{f(u_{n-1})} \frac{r_{n+1} \Delta z_{n+1}}{f(u_{n-1})} \frac{\Delta f(u_{n-1})}{f(u_{n-1}) f(u_{n+1-1})} \leq -q_n.
\]

Summing the above inequality from \(n_1\) to \(n\), we get
\[
w_{n+1} < w_{n_1} - \sum_{i=n_1}^{n} q_i
\]
and, by (10), we see that \(w_n \to -\infty\) as \(n \to \infty\). Then (11) implies that \(u_n < 0\) eventually.

Now we observe that from (10) it follows there exists \(n_2 \geq n_1\) sufficiently large such that
\[
\sum_{i=n_2}^{n} q_i \geq 0 \quad \text{for} \quad n \geq n_2.
\]
(12)

Also we may assume that \(u_{n-1} < 0\) and \(\Delta u_{n-1} > 0\) for \(n \geq n_2\). Summing up both sides of (1) from \(n_2\) to \(n\) we have
\[
\sum_{i=n_2}^{n} \Delta (r_n \Delta z_n) = -\sum_{i=n_2}^{n} q_i f(u_{i-1})
\]
and according to summation by parts formula we may write
\[
r_{n+1} \Delta z_{n+1} - r_{n_2} \Delta z_{n_2} = -f(u_{n-1}) \sum_{i=n_2}^{n} q_i + \sum_{i=n_2}^{n-1} \Delta f(u_{i-1}) \sum_{j=n_2}^{i} q_j.
\]

By the assumptions, from the above equality we get
\[
r_n \Delta z_{n+1} \geq r_{n_2} \Delta z_{n_2} > 0,
\]
which implies
\[
z_{n+2} \geq z_{n+1} + r_{n_2} \Delta z_{n_2} \sum_{i=n_2}^{n} \frac{1}{r_{i+1}} \to \infty \quad \text{as} \quad n \to \infty,
\]
but this contradicts the fact that \((z_n)\) is eventually negative. The case that \((\Delta u_n)\) is eventually negative can be treated in a similar fashion and so the proof is completed.

\textit{Remark 3.} Theorem 3 extends Theorem 2 of [18] and Theorem 3 of [13].

\textbf{References}


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