ON THE RATE OF POINTWISE CONVERGENCE OF THE KANTOROVICH-TYPE OPERATORS

ABSTRACT: For bounded or some locally bounded functions \( f \) measurable on an interval \( I \) there is estimated the rate of convergence of the Kantorovich-type operators \( L_n f \) at those points \( x \in \text{Int} I \) at which the one-sided limits \( f(x \pm 0) \) exist. In the main theorems the Chanturiya modulus of variation is used.

KEY WORDS: Kantorovich-type operator, rate of convergence, modulus of variation.

1. PRELIMINARIES

Let \( I \) be a finite or infinite interval and let \( M(I) \) be the class of all measurable complex-valued functions bounded on \( I \). In the case when \( I \) is an infinite interval, denote by \( M_{\text{loc}}(I) \) the class of all functions measurable on \( I \) and bounded on every compact subinterval of \( I \). Given any \( n \in \mathbb{N} := \{1,2,\ldots\} \), let \( J_n \) be a set of indices contained in \( Z := \{0, \pm 1, \pm 2, \ldots\} \) and let \( I \) be the union of non-overlapping intervals \( I_{j,n} \), \( (j \in J_n) \), with increasing left [right] end points. Introduce, formally, for functions \( f \) belonging to \( M(I) \) or \( M_{\text{loc}}(I) \), the discrete operators \( L_n \) defined by

\[
L_n f(x) := \sum_{j \in J_n} f(\xi_{j,n}) p_{j,n}(x) \quad (x \in I, \ n \in \mathbb{N}),
\]

where \( \xi_{j,n} \in I_{j,n} \) and \( p_{j,n} \) are non-negative functions continuous on \( I \). Denote by \( L_n^* \) the Kantorovich-type modification of operators (1), given by

\[
L_n^* f(x) := \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} f(t) \, dt \quad (x \in I, \ n \in \mathbb{N}),
\]

with \( m_{j,n} = \text{meas } I_{j,n} \). Assume that, for every \( x \in I \),

\[
\rho_n(x) := \sum_{j \in J_n} p_{j,n}(x) - 1 \to 0 \quad \text{as } n \to \infty,
\]

and that

\[
\mu_{2,n}^*(x) := \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} (t - x)^2 \, dt < \infty \quad (n \in \mathbb{N}).
\]
In view of the Shisha and Mond Theorem ([4], pp. 28-29) we have

$$\lim_{n \to \infty} L_n^* f(x) = f(x)$$

at every point $x$ of continuity of $f \in M(I)$ at which $\mu^*_n(x) \to 0$ as $n \to \infty$. Some approximation properties of certain concrete operators of the form (2) for continuous or Lebesgue-integrable functions $f$ are investigated e.g. in [5, Chap.9], [9, Chap. II].

In this paper we present general quantitative inequalities for the rate of pointwise convergence of $L_n^* f(x)$ for functions $f \in M(I)$ (or $f \in M_{loc}(I)$) at those points $x \in \text{Int } I$ at which the one-sided limits $f(x \pm 0)$ exist. In particular, inequalities of this type for the Bernstein-Kantorovich polynomials are obtained in [10]. Analogous results for operators (1) are given in [1].

For the sake of brevity we write

$$s(x) := \frac{1}{2} \{f(x+0) + f(x-0)\}, \quad r(x) := \frac{1}{2} \{f(x+0) - f(x-0)\}.$$ 

Our main estimates concerning the deviation $|L_n^* f(x) - s(x)|$ are expressed in terms of the modulus of variation of the function

$$g_x(t) := \begin{cases} f(t) - f(x+0) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-0) & \text{if } t < x, \end{cases} \quad (t \in I).$$

Given any positive integer $k$, the modulus of variation $v_k(g;Y)$ of a bounded function $g$ on a finite or infinite interval $Y$ is defined as the upper bound of the set of all numbers

$$\sum_{j=1}^{k} |g(b_j) - g(a_j)|$$

over all systems $\Pi_k$ of $k$ non-overlapping intervals $(a_j, b_j)$ contained in $Y$. If $k = 0$ we take $v_0(g;Y) = 0$. Clearly, $v_k(g;Y)$ is a non-decreasing function of $k$. Some basic properties of this modulus can be found e.g. in [3].

In our considerations we use the standing notation:

$$I_x(h) := [x+h, x] \cap I \quad \text{if } h < 0, \quad I_x(h) := [x, x+h] \cap I \quad \text{if } h > 0,$$

$$J_x(h) := [x-h, x+h] \cap I \quad \text{for } h > 0.$$ 

The integral part of a real number $u$ is denoted by $[u]$. 
2. MAIN RESULTS

Let us note that under the assumption $f \in M(I)$ the operators (2) can be written in the form

\[ L_n^* f(x) = \int f(t) H_n(x,t) dt \]

with

\[ H_n(x,t) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \chi_{j,n}(t), \]

where $\chi_{j,n}$ denotes the characteristic function of the interval $I_{j,n}$. The same is also true for $f \in M_{loc}(I)$, satisfying the suitable growth condition (as in Theorem 2 below).

Consider a point $x \in \text{Int} I$ at which both limits $f(x \pm 0)$ exist. It is clear that

\[ L_n^* f(x) - s(x) = L_n^* g_x(x) + r(x) L_n^* \text{sgn}_x(x) + s(x) \rho_n(x), \]

where

\[ \text{sgn}_x(t) := \begin{cases} 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x. \end{cases} \]

In order to evaluate the term $L_n^* g_x(x)$ it is convenient to use the representation (5) and write

\[ L_n^* g_x(x) = \left( \int_{I_x(-a)} + \int_{I_x(b)} \right) g_x(t) H_n(x,t) dt + g_x(a,b) \int_{D_x(a,b)} g_x(t) H_n(x,t) dt \]

where $a > 0$, $b > 0$, $D_x(a,b) = I[x-a,x+b]$, $g_x(a,b) = 0$ if neither of the points $x-a$, $x+b$ belongs to Int $I$, and $g_x(a,b) = 1$ otherwise.

**Lemma.** Suppose that $x \in \text{Int} I$ and that $f$ is bounded on an interval $I_x(h)$, $h \neq 0$. Choose a positive null sequence $(d_n)_n$ such that $d_n \leq 1/2$ and write $\mu := [1/d_n]$. Then, for every $n \in N$,

\[ \left| \int_{I_x(b)} g_x(t) H_n(x,t) dt \right| \leq \]
\[ \leq P_n(x, h) \left\{ \sum_{i=1}^{\mu-1} \frac{1}{i^2} v_i(g_x; I_x(i h d_n)) + \frac{1}{\mu^2} v_{1} (g_x; I_x(h)) \right\}, \]

where \( P_n(x, h) := 1 + \rho_n(x) + 8 \mu_{2,n}(x) h^{-2} d_n^{-2} \).

This result follows by the same method as in [1] or [2], and we omit the details.

If the function \( f \) is bounded on \( I \) and if at least one of the points \( x - a, x + b \) belongs to \( \text{Int} \ I \), then the obvious inequality

\[
\int_{|t-x| \leq s} H_n(x, t) \, dt \leq \frac{1}{s^2} \mu_{2,n}^*(x), \quad (x \in I, \ s > 0)
\]

yields the estimate

\[
(9) \quad \left| \int_{D_x(a, b)} g_x(t) H_n(x, t) \, dt \right| \leq \frac{1}{c^2} \mu_{2,n}^*(x) v_1(g_x; I),
\]

where \( \mu_{2,n}^*(x) \) is defined by (4) and \( c = \min \{a, b\} \).

Taking into account identities (6), (7) with \( a = b = 1 \), inequality (9) and the Lemma (with \( h = -1 \) and \( h = 1 \)), we can state our main result as follows.

**Theorem 1.** Suppose that, for all \( x \in I \) and all \( n \in N \),

\[
(10) \quad \sum_{j \in J_n} P_{j,n} (x) \equiv 1 + \rho_n (x) \leq \varphi_1 (x),
\]

\[
(11) \quad \mu_{2,n}^*(x) \leq \varphi_2 (x) d_n^2,
\]

where \( \varphi_1, \varphi_2 \) are some positive functions continuous on \( I \) and \( (d_n)_{1}^{\infty} \) is a positive null sequence. If \( f \in M(I) \) and if at a point \( x \in \text{Int} \ I \) the one-sided limits \( f(x \pm 0) \) exist then, for all positive integers \( n \) such that \( d_n \leq 1/2 \),

\[
|L_n^* f(x) - s(x)| \leq P(x) \left\{ \sum_{i=1}^{\mu-1} \frac{1}{i^2} v_i(g_x; J_x(id_n)) + \frac{1}{\mu^2} v_{1} (g_x; J_x(1)) \right\} + \\
+ \partial_x (1, 1) \varphi_2 (x) d_n^2 v_1(g_x; I) + \left| r(x) L_n^* \text{sgn}_x (x) \right| + |s(x) \rho_n (x)|,
\]

where \( \mu = [1/d_n] \), \( P(x) := 2 \varphi_1 (x) + 8 \varphi_2 (x) \) and \( \rho_n (x) \) is defined by (3).
Following [1, Theorem 2] (or [2, Theorem 2]) one can get the result for unbounded functions $f$ on infinite interval $I$.

**Theorem 2.** Let $I = [0, \infty)$ or $I = (-\infty, \infty)$ and let conditions (10), (11) be fulfilled. Suppose that a function $f$ of class $M_{loc}(I)$ satisfies the growth condition

$$|f(x)| \leq \psi(x) \quad (x \in I)$$

with a positive continuous function $\psi$ such that for all $n \geq n_0 \in N, \ x \in I$

$$\sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} \psi^2(t) \, dt \leq \varphi_3(x),$$

$0 < \varphi_3(x) < \infty$. If at a point $x \in \text{Int} I$ the limits $f(x \pm 0)$ exist and if $A$ is an arbitrary positive number for which $|x| \leq A$ then, for every integer $n \geq n_0$, such that $d_n \leq 1/2$, we have

$$|L_{n}^* f(x) - s(x)| \leq$$

$$\leq 2P(x,A)\left\{\sum_{i=1}^{\mu-1} \frac{1}{i!^2} v_i\left(g_x;J_x(A_d_n)\right) + \frac{1}{\mu^2} v_{\mu}\left(g_x;J_x(A)\right)\right\} +$$

$$+ \Lambda(x,A)d_n + |r(x)L_{n}^* \text{sgn}_x(x)| + |s(x)\rho_n(x)|,$$

where

$$P(x,A) := \varphi_1(x) + 8\varphi_2(x)/A^2,$$

$$\Lambda(x,A) := A^{-1}(\varphi_2(x)\varphi_3(x))^{1/2} + \frac{1}{4} A^{-2} \psi(x)\varphi_2(x)$$

and the remaining quantities are of the same meaning as in Theorem 1.

Now, let us denote by $BV_p(I)$ $(1 \leq p < \infty)$ the class of all functions of bounded $p$-th power variation on $I$. Here, by $p$-th power variation of a function $g$ on the interval $Y \subseteq I$ we will mean the upper bound of the set of non-negative numbers

$$\left\{\sum_j |g(b_j) - g(a_j)|^p\right\}^{1/p}$$

over all finite systems of non-overlapping intervals $(a_j, b_j) \subseteq Y$. We will denote it by $V_p(g;Y)$. Clearly, if $V_p(g;Y) < \infty$ then for every positive integer $j$,
\[ v_j(g;Y) \leq j^{1-1/p} V_p(g;Y). \]

Using this inequality and proceeding similarly to [11, pp.152-153] we get from Theorem 1 the following

**Corollary.** Suppose that conditions (10), (11) are satisfied. If \( f \in BV_p(I) \) then, for all \( x \in \text{Int} I \) and \( n \in \mathbb{N} \) such that \( 0 < d_n \leq 1/2 \),

\[
|L_n^* f(x) - s(x)| \leq Q(x) \frac{1}{\mu^{1+1/p}} \sum_{k=0}^{\mu^2-1} \frac{1}{(\sqrt{k+1})^{1-1/p}} V_p(g_x;Y_k) + \\
+ |r(x)L_n^* \text{sgn}_x(x)| + |s(x)\rho_n(x)|,
\]

where \( Y_k = J_{x}(1/\sqrt{k}) \) if \( k = 1,2,\ldots,\mu^2-1 \), \( Y_0 = I \), \( Q(x) := 15(\varphi_1(x) + 8\varphi_2(x)) \), \( \mu, L_n^* \text{sgn}_x(x), \rho_n(x) \) have the same meaning as in Theorem 1.

In the case of unbounded functions \( f \) Theorem 2 leads to the analogous Corollary, too.

**Remark 1.** The term \( L_n^* \text{sgn}_x(x) \) occurring in our estimates can be written in the form

\[ L_n^* \text{sgn}_x(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left( \int_{t>x} \chi_{j,n}(t)dt - \int_{t<x} \chi_{j,n}(t)dt \right). \]

Suppose that \( x \) belongs to the interval \( I_{i,n} = [\alpha_{i,n}, \beta_{i,n}] \). Then

\[ L_n^* \text{sgn}_x(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left\{ \beta_{i,n} \int_x^{\beta_{i,n}} \chi_{j,n}(t)dt + \sum_{k \geq 1} \int_{I_{k,n}} \chi_{j,n}(t)dt \right\} + \\
- \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left\{ \sum_{k \leq 1} \int_{I_{k,n}} \chi_{j,n}(t)dt + \int_{a_{i,n}}^{\chi} \chi_{j,n}(t)dt \right\}. \]

(It is understood that the summation in the inner sums is extended over \( k \in J_n \).) Further,
\[ L_n^* \text{sgn}_x(x) = (m_{i,n})^{-1} p_{i,n}(x)(\beta_{i,n} - x) - (m_{j,n})^{-1} p_{j,n}(x)(x - \alpha_{i,n}) + \]
\[ + \sum_{j<l} (m_{j,n})^{-1} p_{j,n}(x)m_{j,n} - \sum_{j<l} (m_{j,n})^{-1} p_{j,n}(x)m_{j,n} = \]
\[ = (m_{i,n})^{-1} p_{i,n}(x)(\beta_{i,n} - 2x + \alpha_{i,n}) + \sum_{j<l} p_{j,n}(x) - \sum_{j<l} p_{j,n}(x). \]

Consequently,
\[ |L_n^* \text{sgn}_x(x)| \leq p_{i,n}(x) + \left| \sum_{j<l} p_{j,n}(x) - \sum_{j<l} p_{j,n}(x) \right|. \]

The above estimate is useful in applications.

Remark 2. In view of the continuity of the function \( g_x \) at \( x \), the right-hand side of inequality (8) converges to zero as \( n \to \infty \) (see e.g. Remark 1 in [11]). Moreover, for many operators of the form (2), condition (3) is satisfied and
\[ \lim_{n \to \infty} L_n^* \text{sgn}_x(x) = 0 \text{ at every } x \in \text{Int } I. \]

Consequently, for these operators, the right-hand sides of the inequalities given in Theorems 1, 2 and Corollary converge to zero as \( n \) tends to infinity.

3. EXAMPLES

Let \( \{\xi_k\}_{i=1}^\infty \) be a sequence of independent and identically distributed random variables with expectations \( E\xi_k = x \) and finite variances \( E(\xi_k - E\xi_k)^2 = \sigma^2(x) \), where \( x \) is a real parameter taking values in an interval \( I \subseteq [0, \infty) \). Suppose that \( \xi_1 \) has the lattice distribution \( F := \{p_{j,n}(x) : x \in I, j \in J_1\} \) concentrated on a set \( J_1 \subseteq N_0 := \{0,1,2,\ldots\} \). The operators (1) with the system \( \{p_{j,n}(x) : x \in I, j \in J_1\} \) being the \( n \)-fold convolution of \( F \) with itself and \( \xi_{j,n} = j/n \) are called the discrete Feller operators ([6], p. 218). Consider the corresponding Kantorovich-type operators \( L_n^* \) defined by (2) in which \( \xi_{j,n} \in J_{i,n} \) and \( m_{j,n} \leq 1/n \) for all \( j \in J_{i,n}, n \in N \). Suppose that \( x \in I_{i,n} \) with some index \( l \in J_{i,n} \) and put \( \lambda = (l - nx)/\sqrt{n\sigma(x)} \). Then
\[ \left| \sum_{j<l} p_{j,n}(x) - \frac{1}{2} \right| \leq \left| \sum_{j<l} p_{j,n}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\lambda \exp(-u^2/2)du \right| + \]
\[
\frac{1}{\sqrt{2\pi}} \left| \int_0^1 \exp(-u^2/2) \, du \right|.
\]

If, moreover,
\[
\sigma^2(x) > 0 \quad \text{and} \quad \beta(x) := E(|\zeta_1 - x|^3) < \infty
\]
then, in view of the Berry-Esséen Theorem ([6], p. 515),
\[
\left| \sum_{j \leq l} p_j(x) - \frac{1}{2} \right| \leq \frac{\tau \beta(x)}{\sqrt{n} \sigma^3(x)} + \frac{1}{\sqrt{2\pi n} \sigma(x)}
\]
and
\[
p_{l,n}(x) \leq \frac{2 \tau \beta(x)}{\sqrt{n} \sigma^3(x)} + \frac{1}{\sqrt{2\pi n} \sigma(x)},
\]
where \((2\pi)^{-1/2} < \tau < 0.82\) (see e.g. [2], p. 101). Applying (14) we get
\[
|L_n^* \operatorname{sgn}_x(x)| \leq 2 \left( \left( \frac{1}{2} - \sum_{j \leq l} p_j(x) \right) + p_{l,n}(x) \right) \leq \frac{2}{\sqrt{n}} \left( \frac{3 \tau \beta(x)}{\sigma^3(x)} + \frac{2}{\sqrt{2\pi} \sigma(x)} \right).
\]

Now, we present an application of our Theorems to some concrete Feller-Kantorovich operators.

1. The Bernstein-Kantorovich polynomials \(B_n^* = L_n^*\) are defined by (2) with
\[
p_j(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad I_{j,n} = \frac{j}{n+1} \frac{j+1}{n+1}, \quad x \in I = [0,1], \quad j \in J_n = \{0,1,\ldots,n\}
\]
and \(m_{j,n} = 1/(n+1)\). In this case, \(\sigma^2(x) = x(1-x), \quad \beta(x) = x(1-x) \cdot (2x^2 - 2x + 1)\) ([8], p. 98). Moreover,
\[
\mu_{2,n}(x) = \frac{3x(1-x)(n-1)+1}{3(n+1)^2} \quad \text{for all} \quad x \in I, \quad n \in N;
\]
whence \(\mu_{2,n}(x) \leq 1/4n\) for all \(n \in N\) and \(\mu_{2,n}(x) \leq 3x(1-x)/n\) for \(n \geq (x(1-x))^{-1}\). Consequently, Theorem 1 and Corollary apply for \(x \in (0,1)\) and all \(n \geq 2\), with \(d_n = 1/\sqrt{n}, \quad \varphi_1(x) = 1, \quad \varphi_2(x) = 1/4, \quad \varphi_3(x), \varphi_4(x) = 0, \quad \rho_n(x) = 0\) and, in view of (15),
\[
|B_n^* \operatorname{sgn}_x(x)| \leq 10(2x^2 - 2x + 1)/\sqrt{nx(1-x)}.
\]
For \(n \geq (x(1-x))^{-1}\) it is convenient to choose \(\varphi_2(x) = 3x(1-x)\).
2. Let $S_n^* = I_n^*$ be the modified Szasz-Mirakyan operators defined by (2) with $p_{j,n}(x) = (nx)^je^{-nx}/j!, \ I_{j,n} = \left[ \frac{j}{n}, \frac{j+1}{n} \right], \ x \in I = [0, \infty), \ j \in J_n = N_0$ and $m_{j,n} = 1/n$. In this case, $\sigma^2(x) = x$, $\beta(x) \leq 8x^3 + 6x^2 + x$ ([8], p. 99), $\mu_{2,n}^*(x) = x/n + 1/3n^2$ for all $x \in I, \ n \in N$. Consequently, Theorem 1 and Corollary apply for $x > 0$ and all $n \geq 2$ with $d_n = 1/\sqrt{n}$, $\varphi_1(x) = 1$, $\varphi_2(x) = (6x + 1)/12$, $\vartheta_2(1,1) = 1$, $\rho_n(x) = 0$, and

$$|S_n^* \text{sgn}_x(x)| \leq 10(4x^2 + 3x + 1)/\sqrt{n}x,$$

by (15).

3. The Baskakov-Kantorovich operators $U_n^* = I_n^*$ are defined by (2) in which $p_{j,n}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j}, \ I_{j,n} = \left[ \frac{j}{n}, \frac{j+1}{n} \right], \ x \in I = [0, \infty)$, $j \in J_n = N_0$, $m_{j,n} = 1/n$. Now $\sigma^2(x) = x(1+x)$, $\beta(x) \leq 16x^3 + 9x^2 + x$ ([8], p. 100). It is easy to see that

$$\mu_{2,n}^*(x) = \frac{x(x+1)}{n} + \frac{1}{3n^2} \quad \text{for all} \quad x \in I, \ n \in N.$$

Hence, our results can be applied for $x > 0$ and all $n \geq 2$, with $d_n = 1/\sqrt{n}$, $\varphi_1(x) = 1$, $\varphi_2(x) = (1/3)(3x(x+1)+1)$, $\vartheta_2(1,1) = 1$, $\rho_n(x) = 0$. In this case inequality (15) implies

$$|U_n^* \text{sgn}_x(x)| \leq 10(8x^2 + 5x + 1)/\sqrt{nx(x+1)^3}.$$

Finally, let us consider the generalized Favard operators $F_n^* = I_n^*$, which are not the Feller-type ones. They are defined by formula (1) in which $\xi_{j,n} = j/n$, $j \in J_n = Z$, $x \in I = (-\infty, \infty)$ and

$$p_{j,n}(x) = p_{j,n}(\gamma, x) = (\sqrt{2\pi n \gamma_n})^{-1} \exp \left(-\frac{1}{2} \gamma_n^{-2} \frac{j-x}{n} \right),$$

where $\gamma = (\gamma_n)_{n=1}^\infty$ is a positive null sequence such that

$$n^2 \gamma_n^2 \geq \frac{1}{2} \pi^{-2} \log n \quad \text{for} \quad n \geq 2, \quad \gamma_1^2 \geq \frac{1}{2} \pi^{-2} \log 2.$$
(see [7]). Denote by $F_n^*$ their Kantorovich modification of the form (2) with
$I_{j,n} = [j/n, (j+1)/n]$, and $m_{j,n} = 1/n$ for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. As is known
([7]), for all $x \in I$ and $n \in \mathbb{N}$
\[ |\rho_n(x)| \leq 2 \quad \text{or} \quad |\rho_n(x)| \leq 7\pi \gamma_n \]
and
\[ \mu_{2,n}(x) := \sum_{j=-\infty}^{\infty} \left( \frac{j}{n} - x \right)^2 p_{j,n}(x) \leq 51 \gamma_n^2. \]
An easy computation shows that
\[ \mu_{2,n}^*(x) \leq \mu_{2,n}(x) + \frac{1}{n} \sqrt{\mu_{2,n}(x)} \sqrt{1 + \rho_n(x)} + \frac{1}{3n^2} (1 + \rho_n(x)) \leq 158 \gamma_n^2. \]
Hence, applying Theorem 1 (or Corollary) to these operators, we can put
$\varphi_1(x) = 3$, $\varphi_2(x) = 158 \kappa^2$ and $d_n = \gamma_n / \kappa$, where $\kappa := \max \{1, 2^{\sup} \gamma_j \}$. In order

Hence, applying Theorem 1 (or Corollary) to these operators, we can put
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In order to estimate the term $F_n^* \text{sgn}_x (x)$ we use inequality (14) and, arguing similarly to [1, Sect. 4.1], we obtain
\[ |F_n^* \text{sgn}_x (x)| \leq 3(\sqrt{2\pi n} \gamma_n)^{-1} \leq 3\sqrt{\pi (\log n)}^{-1/2} \quad \text{for} \quad n \geq 2. \]
The same estimates can also be used in Theorem 2. Additionally, let us note that

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condition (13) is fulfilled for $q \gamma_n^2 \leq 3/128$ with $\varphi_1(x) = c(q) \exp(4q x^2)$, $c(q)$

being some positive constant depending on $q$.

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