TINGFU WANG, DONGHAI JI AND MINLI LI

U-POINTS AND QUASI-U-POINTS IN ORLICZ FUNCTION SPACES

ADSTRACT: In this paper, notions of U-points and quasi-U-points are defined. There is given a characterization of U-point and quasi-U-point in Orlicz function spaces equipped with the Luxemburg norm.

KEY WORDS: Orlicz function space, U-point, quasi-U-point, locally U-space, quasi-U-space

1. INTRODUCTION

The concept of U-spaces was introduced by Gao Ji and Lau Kasing in 1991 (see [3]). U-spaces have uniformly normal structure and many other interesting properties. In this paper we introduce the notation U-points and quasi-U-points. That kind of points are necessary to define local U-spaces and quasi-U-spaces. We give a complete characterization of U-points and quasi-U-points in Orlicz function spaces with the Luxemburg norm. Basing on that characterization, we obtain some criteria for an Orlicz function space to be the local-U-space and quasi-U-space.

Let \( S(X) \) be the unit sphere of a Banach space \( X \). A point \( x \in S(X) \) is called an U-point provided for any sequence \( \{x_n\}_{n=1}^{\infty} \subset S(X) \), satisfying \( \|x_n + x\| \to 2 \) we have \( f(x_n - x) \to 0 \) uniformly for all \( f \in \nabla_x \), where \( \nabla_x \) denotes the set of all supporting functionals of \( x \). A point \( x \in S(X) \) is called a quasi-U-point if for any \( y \in S(X) \), such that \( \|x + y\| = 2 \), we have \( f(y - x) = 0 \) for all \( f \in \nabla_x \). Obviously, every U-point is a quasi-U-point. A Banach space \( X \) is said to be a local (resp. quasi-) U-space if each point \( x \in S(X) \) is an U (resp. a quasi-U)-point.

Denote by \( \mathbb{N} \) and \( \mathbb{R} \) the sets of natural and real numbers, respectively. Let \( (G, \Sigma, \mu) \) be a measure space with a finite and atomless measure \( \mu \). Denote by \( L^0 \) the set of all \( \mu \)-equivalence classes of real valued measurable functions defined on \( G \).

A map \( M : \mathbb{R} \to [0, \infty) \) is said to be an Orlicz function if \( M \) is vanishing at 0, even, convex and not identically equal to 0.

Denote by \( p(u) \) and \( p_-(u) \) the right and left derivative of \( M(u) \), respectively. An Orlicz function \( M \) is called an N-function if
\[
\lim_{u \to +\infty} \frac{M(u)}{u} = \infty \quad \text{and} \quad \lim_{u \to 0} \frac{M(u)}{u} = 0.
\]

By the Orlicz function space \( L_M \) we mean
\[
L_M = \left\{ x \in L^0 : \rho_M(cx) = \int_G M(cx(t))d\mu < \infty \quad \text{for some} \quad c > 0 \right\}
\]
equipped with so called the Luxemburg norm
\[
\| x \| = \inf \left\{ \varepsilon > 0 : \rho_M \left( \frac{x}{\varepsilon} \right) \leq 1 \right\}
\]
or with equivalent
\[
\| x \|_0 = \inf_{k > 0} \frac{1}{k} (1 + \rho_M (kx))
\]
called the Orlicz norm. To simplify denotations, we put \( L_M = (L_M, \| \cdot \|) \) and \( L^0_M = (L_M, \| \cdot \|_0) \).

For every Orlicz function \( M \) we define the complementary function \( N : R \to [0, \infty) \) by the formula
\[
N(v) = \sup_{u > 0} \{ u \mid v \leq M(u) \}
\]
for every \( v \in R \). The complementary function \( N \) is also an Orlicz function.

We say that the Orlicz function \( M \) satisfies the \( \Delta_2 \)-condition if there exist a constant \( k \geq 2 \) and \( u_0 > 0 \) such that
\[
M(2u) \leq kM(u)
\]
for every \( |u| \geq u_0 \).

We say that the Orlicz function \( M \) satisfies the \( \Delta_2 \)-condition if its complementary function \( N \) satisfies the \( \Delta_2 \)-condition.

We say that the Orlicz function \( M \) is strictly convex (write \( M \in SC[0, \infty) \)) if for any \( u \neq v \) and \( \alpha \in (0,1) \) we have
\[
M(\alpha u + (1 - \alpha)v) < \alpha M(u) + (1 - \alpha)M(v).
\]

For more details we refer to [5].

An interval \([a, b]\) is called a structural affine interval of \( M \), or simply SAI of \( M \), provided that \( M \) is affine on \([a, b]\) and it is not affine on either \([a - \varepsilon, b]\) or \([a, b + \varepsilon]\) for any \( \varepsilon > 0 \). Let \([a_i, b_i]\) (\( i = 1, 2, \ldots \)) be all the SAI of \( M \). Denote by
$S_M = R \setminus \bigcup_i (a_i, b_i)$ the set of strictly convex points of $M$. Moreover, for any $x \in L_M$ we define

$$\Theta(x) = \inf \left\{ c > 0 : \rho_M \left( \frac{x}{c} \right) < \infty \right\},$$

$$d(x, E_M) = \inf \{ \| x - y \| : y \in E_M \}$$

and for any $v \in L_N^0$,

$$K(v) = \left\{ k > 0 : \| v \|^0 = \frac{1}{k} (1 + \rho_N(kv)) \right\}.$$

**Lemma 1.** $f = v + \phi \in L_M^*$ is a supporting functional of $x \in S(L_M)$ iff

1. $\rho_M(x) = 1$,
2. $\| \phi \| = \phi(x)$,
3. $v(t)x(t) \geq 0$ and $p_-(\| x(t) \|) \leq k \| v(t) \| \leq p(\| x(t) \|)$, for $\mu$-a.e. $t \in G$,

where $v \in L_N^0$, $\phi \in L_M^*$ is a singular functional and $k \in K(v)$.

**Proof:** The proof of this Lemma can be found in [10].

**Lemma 2.** If $x \in S(L_M)$ and there is $\tau > 0$ such that $\rho_M((1+\tau)x) < \infty$, then

1. $\rho_M(x) = 1$,
2. all supporting functionals of $x$ belong to $L_N^0$,
3. for any $k \in K(v)$ we have $\lim_{\mu \to 0} \sup_{v \in \nabla_k} \rho_N(kv) = 0$ ($e \in \Sigma$),
4. $\rho_N(p(x)) < \infty$,
5. for any sequence $\{x_n\}_{n=1}^\infty \subset S(L_M)$ such that $\| x_n + x \| \to 2$, we have $\rho_M(x_n) \to 1$ and $\rho_M((x_n + x)/2) \to 1$,
6. for any $y \in S(L_M)$ with $\| x + y \| = 2$, we have $\rho_M(y) = \rho_M\left( \frac{x + y}{2} \right) = 1$.

**Proof:** (i) is obvious.

(ii) It is easy to see that $\nabla_x \subset S(L_N^0)$ for any $x \in E_M$. Suppose that $x \in L_M \setminus E_M$ has a supporting functional of the form $f = v + \phi$, where $v \in L_N^0$ and $\phi \in L_M^*$ is a singular functional. Then, by Lemma 5.3 in [10], we get
\[ \phi(x) = \|\phi\| = \sup_{z \in L_M \cdot E_M} \frac{\phi(z)}{\Theta(z)} \geq \frac{\phi(x)}{\Theta(x)} > \phi(x). \]

A contradiction.

(iii) Otherwise, there is \( \varepsilon > 0 \) and a sequence of measurable sets \( \{e_n\}_{n=1}^\infty \subset G \) with \( e_n \downarrow 0 \) and \( v_n \in \nabla_x \) such that \( \rho_N(k_n v_n|e_n) \geq \varepsilon > 0 \), where \( k_n \in K(v_n) \). Since \( \rho_M((1+\tau)x) < \infty \), there is \( n \in N \) such that \( \rho_N((1+\tau)x|e_n) < \tau\varepsilon/2 \). Hence

\[
0 = k_n \|v_n\|_M^0 - k_n \langle x, v_n \rangle = 1 + \rho_N(k_n v_n) - \langle x, k_n v_n \rangle =
\]

\[
= \int_G (M(x(t)) + N(k_n v_n(t)) - x(t)k_n v_n(t))dt >
\]

\[
> \int_{e_n} \left( M(x(t)) + N(k_n v_n(t)) - \frac{1+\tau}{1+\tau} x(t)k_n v_n(t) \right) dt \geq
\]

\[
\geq \int_{e_n} \left( N(k_n v_n(t)) - \frac{1}{1+\tau} (M((1+\tau)x(t)) + N(k_n v_n(t)) \right) dt =
\]

\[
= \frac{\tau}{1+\tau} \rho_N(k_n v_n|e_n) - \frac{1}{1+\tau} \rho_M((1+\tau)x|e_n) \geq
\]

\[
\geq \frac{\tau\varepsilon}{1+\tau} - \frac{\tau\varepsilon}{2(1+\tau)} = \frac{\tau\varepsilon}{2(1+\tau)},
\]

what is impossible.

(iv) Since

\[
M((1+\tau)u) > \int_u p(s)ds \geq \tau up(u) \geq \tau N(p(u)),
\]

we have

\[
\rho_N(p(x)) < \frac{1}{\tau} \rho_M((1+\tau)x) < \infty.
\]

(v) For any \( \varepsilon > 0 \), a natural number \( n \) can be found such that

\[
\left\| (1+\varepsilon)\frac{x_n+x}{2} \right\| > 1.
\]

Hence

\[
1 < \rho_M\left( (1+\varepsilon)\frac{x_n+x}{2} \right) = \rho_M\left( \frac{1+\varepsilon}{2} x_n + \frac{(1-\varepsilon)(1+\varepsilon)}{2} x \right) \leq
\]
\[
\frac{1+\varepsilon}{2} \rho_M(x_n) + \frac{1-\varepsilon}{2} \rho_M\left(\frac{1+\varepsilon}{1-\varepsilon} x\right) = \\
= \frac{1+\varepsilon}{2} \rho_M(x_n) + \frac{1-\varepsilon}{2} (\rho_M(x) + o(\varepsilon)) = \\
= \frac{1+\varepsilon}{2} \rho_M(x_n) + \frac{1-\varepsilon}{2} (1 + o(\varepsilon)).
\]

Therefore, by the arbitrariness of \( \varepsilon \), we conclude \( \rho_M(x_n) \to 1 \) as \( n \to \infty \).

Analogously, noticing that \( \|\|(x_n + x)/(2) + x\|\| \to 1 \) as \( n \to \infty \), we can prove that \( \rho_M((x_n + x)/2) \to 1 \) as \( n \to \infty \).

(vi) It follows immediately from (v).

**Lemma 3.** Suppose \( M \in \nabla_2 \) and \([a,b]\) is a SAI of \( M \). Then for any \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( ((M(u) + N(v))/2) - M((u + v)/2) < \eta \) and \( v \in [a,b] \) imply \( u \in [a - \varepsilon, b + \varepsilon] \).

**Proof.** See [2].

**Theorem 1.** An element \( x \in S(L^\infty_M) \) is a quasi-U-point iff the following two conditions are satisfied:

1. there exists \( \tau > 0 \) such that \( \rho_M((1+\tau)x) < \infty \),
2. if there exists a SAI \([a,b]\) of \( M \) with \( p_-(a) < p(a) \) such that \( \mu\{t \in G : \{x(t) = a\} > 0\} \), then \( \mu\{t \in G : \{x(t) \in (c,d)\} = 0 \) for any SAI \([c,d]\) of \( M \); if \( p_-(b) < p(b) \) and \( \mu\{t \in G : \{x(t) = b\} > 0 \), then \( \mu\{t \in G : \{x(t) \in [c,d]\} = 0 \) for any SAI \([c,d]\) of \( M \).

**Proof of necessity.** If (1) is not true, then \( \Theta(x) = 1 \). Denote

\[
G_n = \{t \in G : n - 1 \leq |x(t)| < n\}, \quad n = 1,2,\ldots
\]

Then \( G = \bigcup_{n=1}^{\infty} G_n \). Decompose every set \( G_n \) \( (n = 1,2,\ldots) \) into two sets \( G'_n \) and \( G''_n \) such that \( \mu(G'_n) = \mu(G''_n) \). Define

\[
x'(t) = \sum_{n=1}^{\infty} x(t) |_{G'_n}, \quad x''(t) = \sum_{n=1}^{\infty} x(t) |_{G''_n}.
\]

Then \( x = x' + x'' \). Clearly,
\[ \rho_M(x') < \rho_M \left( \frac{x' + x}{2} \right) < \rho_M(x) \leq 1. \]

On the other hand, for any \( \varepsilon \in (0, 1/2) \), fix \( n_0 \in \mathbb{N} \) such that \((n_0 - 1)/n_0 > (1 - 2\varepsilon)/(1 - \varepsilon)\). We have
\[
\rho_M \left( \frac{1}{1 - 2\varepsilon} \frac{x' + x}{2} \right) > \rho_M \left( \frac{x'}{1 - 2\varepsilon} \right) = \sum_{n=1}^{\infty} \int_{G_n} M \left( \frac{x(t)}{1 - 2\varepsilon} \right) dt \geq
\]
\[
\geq \sum_{n=1}^{\infty} \int_{G_n} M \left( \frac{n - 1}{1 - 2\varepsilon} \right) dt = \frac{1}{2} \sum_{n=1}^{\infty} \int_{G_n} M \left( \frac{n - 1}{1 - 2\varepsilon} \right) dt >
\]
\[
> \frac{1}{2} \sum_{n=1}^{\infty} \int_{G_n} M \left( \frac{n - 1}{n} \frac{x(t)}{1 - 2\varepsilon} \right) dt > \frac{1}{2} \sum_{n > n_0} \int_{G_n} M \left( \frac{x(t)}{1 - \varepsilon} \right) dt = \infty.
\]
This shows that
\[ \| x' \| = \left\| \frac{x' + x}{2} \right\| = 1 \]
and \( \Theta(x') = 1 \). Similarly, we can get that
\[ \| x' \| = \left\| \frac{x' + x}{2} \right\| = 1 \]
and \( \Theta(x'') = 1 \). By \( 1 = \Theta(x') = d(x', E_M) \) and Hahn-Banach Theorem, there exists \( \phi \in L^*_M \) such that \( \phi[E_M] = 0 \) and \( \phi \| = \phi(x') = \Theta(x') = 1 \), i.e. \( \phi \) is a singular functional and supporting functional of \( x' \).

Noticing that
\[ \| \phi \|_{\cup_{n=1}^{\infty} G_n} \geq \phi \|_{\cup_{n=1}^{\infty} G_n} (x') = \phi \left( \frac{x'}{\cup_{n=1}^{\infty} G_n} \right) = \phi(x') = 1 = \| \phi \| \]
and
\[ \| \phi \| = \| \phi \|_{\cup_{n=1}^{\infty} G_n} + \| \phi \|_{\cup_{n=1}^{\infty} G_n}, \]
we have
\[ \| \phi \|_{\cup_{n=1}^{\infty} G_n} = 0. \]
Hence
\[ \phi(x) = \phi(x') + \phi(x'') = \phi(x') + \phi \left( \frac{x''}{\cup_{n=1}^{\infty} G_n} \right) = \phi(x') + \phi \|_{\cup_{n=1}^{\infty} G_n} (x'') = \phi(x') = 1 = \| x \|. \]
i.e \( \phi \) is also a supporting functional of \( x \). But \( \| x'' \| = \| (x'' + x)/2 \| = 1 \) and \( \phi(x'') = 0 \), so \( x \) is not a quasi-\( U \)-point. This contradiction proves (1).

If the first part of (2) does not hold, then there exist the SAIIs \([a, b] \) and \([c, d] \) such that
\[
p_-(a) < p(a), \quad \mu \{ t \in G : |x(t)| = a \} > 0
\]
and
\[
\mu \{ t \in G : |x(t)| \in (c, d) \} > 0.
\]
Let
\[
M(u) = p(a)u + B \quad (u \in [a, b]), \quad M(u) = p(c)u + B' \quad (u \in [c, d]).
\]
Pick \( \varepsilon' > 0 \) satisfying
\[
\mu F = \mu \{ t \in G : x(t) \in [c + \varepsilon', d] \} > 0.
\]
Take \( \tilde{E} \subset E \), \( \tilde{F} \subset F \) with \( \mu \tilde{E} + \mu \tilde{F} > 0 \) and choose \( \varepsilon > 0 \) such that \( p(a)\varepsilon = p(c)\varepsilon' \) and \( a + \varepsilon \leq b \). Put
\[
y(t) = x(t)|_{G \cap (\tilde{E} \cup \tilde{F})} + (a + \varepsilon)|_{\tilde{E}} + (x(t) - \varepsilon')|_{\tilde{F}}.
\]
Then
\[
\rho_M(y) = \rho_M(x)|_{G \cap (\tilde{E} \cup \tilde{F})} + M(a)\mu \tilde{E} + p(a)\varepsilon\mu \tilde{F} +
\]
\[
+ \int_{\tilde{F}} M(x(t))dt - p(c)\varepsilon'\mu \tilde{F} = \rho_M(x) = 1
\]
and consequently
\[
\rho_M(\frac{x+y}{2}) = \rho_M(x)|_{G \cap (\tilde{E} \cup \tilde{F})} + M(a)\mu \tilde{E} + p(a)\frac{\varepsilon}{2} \mu \tilde{F} +
\]
\[
+ \int_{\tilde{F}} M(x(t))dt - p(c)\frac{\varepsilon'}{2} \mu \tilde{F} = \rho_M(x) = 1.
\]
Therefore \( \| y \| = \| (x + y)/2 \| = 1 \).

By (iv) of Lemma 2, \( p_-(x(t)) \in L^0_N \). Set \( \nu = p_-(x)/\| p_-(x) \|_N^0 \). Then, by Theorem 18.5 in [5], \( \nu \) is a supporting functional of \( x \), but
\[
\langle x - y, \nu \rangle = \int_{\tilde{F}} \varepsilon' p_-(x(t))dt - \int_{\tilde{E}} \varepsilon p_-(x(t))dt \geq
\]
\[
\geq \varepsilon' p(c)\mu \tilde{F} - \varepsilon p_-(a)\mu \tilde{E} > \varepsilon' p(c)\mu \tilde{F} - \varepsilon p(a)\mu \tilde{E} = 0,
\]
which contradicts that \( x \) is a quasi-\( U \)-point.

Analogously, we can conclude that the second part of the condition (2) holds true.
Proof of sufficiency. Without loss of generality, we may assume \( x(t) \geq 0 \). Let \( y \in S(L_M) \) and \( \| x + y \| = 2 \). Then, by (i) and (vi) of Lemma 2, we have

\[
\rho_M(x) = \rho_M(y) = \rho_M\left(\frac{x + y}{2}\right) = 1.
\]

Consequently,

\[
0 = \frac{\rho_M(x) + \rho_M(y)}{2} - \rho_M\left(\frac{x + y}{2}\right) = 
\int_0^1 \left(M\left(\frac{x(t) + y(t)}{2}\right) - M\left(\frac{x(t) + y(t)}{2}\right)\right) dt.
\]

Hence, by the convexity of \( M \), we get

\[
\frac{M(x(t)) + M(y(t))}{2} = M\left(\frac{x(t) + y(t)}{2}\right) \quad \text{for } \mu - \text{a.e. } t \in G.
\]

This shows that either \( x(t) = y(t) \) or \( x(t), y(t) \) belong to the same SAI of \( M \).

Consider the following three cases:

(I) There exists a SAI \([a, b]\) of \( M \) with \( p_\leq(b) < p(b) \) such that \( \mu\{t \in G : x(t) = b\} > 0 \), and for any SAI \([c, d]\) of \( M \), \( \mu\{t \in G : c \leq x(t) < d\} = 0 \).

If \( x(t) \) does not belong to any SAI of \( M \), then \( x(t) = y(t) \). If \( x(t) \in [c, d] \), then \( x(t) = d \). Moreover \( y(t) \in [c, d] \), so \( y(t) \leq x(t) \). Thus, we get \( M(y(t)) \leq M(x(t)) \) (\( \mu - \text{a.e.} \)). Combining \( \rho_M(y) = 1 = \rho_M(x) \), it follows immediately that \( x(t) = y(t) \) (for \( \mu - \text{a.e. } t \in G \)), i.e. \( y = x \). For any \( f \in L_M^* \), \( f(y - x) = f(0) = 0 \), i.e. \( x \) is a quasi-U-point.

(II) There exists a SAI \([a, b]\) of \( M \) with \( p_\leq(a) < p(a) \) such that \( \mu\{t \in G : x(t) = a\} > 0 \), and for any SAI \([c, d]\) of \( M \), \( \mu\{t \in G : c \leq x(t) < d\} = 0 \).

Analogously to the proof of (I), we conclude that \( x \) is a quasi-U-point.

(III) \( x(t) \) belongs to a SAI \([a, b]\) of \( M \) and \( p_\leq(x(t)) = p(x(t)) \). By (ii) of Lemma 2, any supporting functional \( v \) of \( x \) belongs to \( S(L_N^0) \). If \( x(t) \in [a, b] \), then \( y(t) \in [a, b] \). Set \( k \in K(v) \). Then

\[
k\nu(t) = p_\leq(x(t)) = p(x(t)) = p(a) = p_\leq(b)
\]

If \( y(t) \in [a, b] \), then

\[
k\nu(t) = p(a) = p(y(t))
\]

If \( y(t) = b \), then

\[
k\nu(t) = p_\leq(b) = p_\leq(y(t))
\]
By Lemma 1, \( \nu \) is also a supporting functional of \( y \). Thus
\[
<\nu, y - x> = <\nu, y> - <\nu, x> = \|y\| - \|x\| = 1 - 1 = 0,
\]
i.e. \( x \) is a quasi-\( U \)-point.

**Corollary 1.** \( L_M \) has quasi-\( U \)-property iff

(i) \( M \in \Delta_2 \),

(ii) for any SAI \([a, b]\) of \( M \), \( p_-(a) = p(a) = p_-(b) = p(b) \).

**Proof of necessity.** (i) If \( M \notin \Delta_2 \), then we can construct \( x \in S(L_M) \) with \( \rho_M(x) < 1 \), so for any \( \varepsilon > 0 \), \( \rho_M((1 + \varepsilon)x) = \infty \). By Theorem 1, \( x \) is not a quasi-\( U \)-point. Furthermore, \( L_M \) cannot have quasi-\( U \)-property.

(ii) If there is a SAI \([a, b]\) of \( M \) with \( p_-(a) < p(a) \), then we can fix \( E \subset G \), \( F \subset G \) with \( 0 < \mu E < \mu G \), \( 0 < \mu F < \mu G \), \( E \cap F = \emptyset \), a real numbers \( c \in (a, b) \) and \( d \) such that
\[
M(a)\mu E + M(c)\mu F + M(d)\mu (G \setminus (E \cup F)) = 1.
\]
Define \( x = d|_E + c|_F + d|_{G \setminus (E \cup F)} \) . Then \( \rho_M(x) = 1 \), i.e. \( \|x\| = 1 \). On the other hand
\[
\mu \{t \in G : x(t) = a\} = \mu E > 0
\]
and
\[
\mu \{t \in G : x(t) \in (a, b)\} \geq \mu F > 0.
\]
Thus, by Theorem 1, \( x \) is not a quasi-\( U \)-point and consequently \( L_M \) has not quasi-\( U \)-property.

The equality \( p_-(b) = p(b) \) can be proved analogously.

**Proof of sufficiency.** For any \( x \in S(L_M) \), by \( M \in \Delta_2 \), we have \( \rho_M(1 + \tau)x) < \infty \). (2) from Theorem 1 holds true trivially. Therefore \( x \) is a quasi-\( U \)-point, i.e. \( L_M \) has quasi-\( U \)-property.

**Theorem 2.** An element \( x \in S(L_M) \) is an \( U \)-point iff

(i) there exists \( \tau > 0 \) such that \( \rho_M((1 + \tau)x) < \infty \),

(ii) if there is a SAI \([a, b]\) of \( M \) with \( p_-(a) < p(a) \) such that
\[
\mu \{t \in G : |x(t)| = a\} > 0,
\]
then
\[
\mu \{t \in G : |x(t)| \in (c, d]\} = 0.
\]
if \( p(b) < p(b) \) and \( \mu \{ t \in G : |x(t)| = b \} > 0 \), then
\[
\mu \{ t \in G : |x(t)| \in [c, d] \} = 0,
\]

(iii) \( M \in \nabla_2 \) or \( \mu \{ t \in G : |x(t)| \in (a, b] \} = 0 \) for any \( \text{SAI} [a, b] \) of \( M \).

Proof of necessity. Without loss of generality, we can assume \( x(t) \geq 0 \). Since an \( U \)-point is a quasi-\( U \)-point, by Theorem 1, (i) and (ii) are satisfied. If (iii) is not true, then there exists a \( \text{SAI} [a, b] \) of \( M \) and \( u_n \uparrow \infty \) such that
\[
\mu \{ t \in G : x(t) \in (a, b] \} > 0
\]
and
\[
M \left( \frac{u_n}{2} \right) > \left( 1 - \frac{1}{n} \right) M \left( \frac{u_n}{2} \right).
\]

Fix \( \varepsilon > 0 \) and \( E_n \subset E \) such that
\[
\mu \Delta E = \mu \{ t \in G : x(t) \in [a + \varepsilon, b] \} > 0
\]
and
\[
\int_{E_n} M(x(t) - \varepsilon)dt + M(u_n)\mu(E \setminus E_n) = \int_E M(x(t))dt.
\]

Then \( \mu(E \setminus E_n) \to 0 \). Define
\[
x_n(t) = (x(t) - \varepsilon)|_{E_n} + u_n|_{E \setminus E_n} + x(t)|_{G \setminus E}.
\]

We have \( \rho_M(x_n) = \rho_M(x) = 1 \). Moreover,
\[
\rho_M \left( \frac{x_n + x}{2} \right) = \int_{E_n} M \left( \frac{x(t) - \varepsilon}{2} \right)dt + \int_{E \setminus E_n} M \left( \frac{x(t) + u_n}{2} \right)dt + \int_{G \setminus E} M(x(t))dt \geq
\]
\[
\geq \frac{1}{2} \left( \int_{E_n} M(x(t))dt + \int_{E_n} M(x(t) - \varepsilon)dt \right) +
\]
\[
+ \int_{E \setminus E_n} M \left( \frac{u_n}{2} \right)dt + \int_{G \setminus E} M(x(t))dt \geq
\]
\[ \geq \frac{1}{2} \left( \int_{E_n} M(x(t)) dt + \int_{E_n} M(x(t) - \varepsilon) dt \right) + \\
+ \left(1 - \frac{1}{n}\right) \frac{M(u_n)}{2} \mu(E \setminus E_n) + \int_{G \setminus E} M(x(t)) dt \]
\[ \geq \frac{1}{2} \left(1 - \frac{1}{n}\right) (\rho_M(x_n) + \rho_M(x)) \to 1. \]

Hence
\[ \| x_n + x \| \to 2. \]

Choose the supporting functional \( y = \frac{1}{k} p(x) \) of \( x \), where \( k = \| p(x) \|_{N_\varepsilon} \). Notice that
\[ \int_{E_n} \varepsilon y(t) dt \geq \frac{1}{4} \varepsilon p(a) \mu E_n \to \frac{\varepsilon}{k} p(a) \mu E \]
\[ \int_{E_n} x(t) y(t) dt \to 0 \]
and
\[ \int_{E_n} u_n y(t) dt \leq \| x_n \|_{E \setminus E_n} \| y \|_{E \setminus E_n} \| \nu \|_{N} \leq \| x_n \| \| y \|_{E \setminus E_n} \| \nu \|_{N} \to 0. \]

Then
\[ \int_{G} (x(t) - x_n(t)) y(t) dt = \int_{E_n} \varepsilon y(t) dt + \int_{E \setminus E_n} (x(t) - u_n) y(t) dt \geq \\
\geq \frac{1}{k} \varepsilon p(a) \mu E_n + \int_{E \setminus E_n} x(t) y(t) dt - \int_{E \setminus E_n} u_n y(t) dt \to \frac{\varepsilon}{k} p(a) \mu E > 0, \]
which contradicts to the fact that \( x \) is an \( U \)-point.

\textbf{Proof of sufficiency.} Let \( x_n \in S(L_M) \) and \( \| x_n + x \| \to 2 \). Without loss of generality assume that \( x(t) \geq 0 \). In the following we will prove the sufficiency considering three cases:

(1) \[ \mu \{ t \in G : x(t) \in (a, b) \} = 0 \] for any SAI \([a, b]\) of \( M \).

Denote
\[ A = \left\{ t \in G : x(t) \in \bigcup_{i=1}^{\infty} \{ a_i \} \right\}. \]

First we prove that
\[ x_n(t) \xrightarrow{\mu} x(t) \text{ on } G \setminus A. \]

Otherwise, there exist \( \varepsilon, \delta > 0 \) such that
\[
\mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon\} \geq \delta.
\]

By Lemma 2, we have \( \rho_M(x) = 1, \rho_M(x_n) \to 1 \) and \( \rho_M\left(\frac{x_n + x}{2}\right) \to 1 \). Since
\[
1 - \rho_M(x_n) = \int_G M(x_n(t)) dt \geq \int_{\{t \in G : |x_n(t)| > D\}} M(x(t)) dt \\
> M(D) \mu\{t \in G : |x_n(t)| > D\}
\]
for every \( D > 0 \), we have
\[
\mu\{t \in G : |x_n(t)| > D\} < \frac{1}{M(D)}.
\]

Thus we may pick \( D \) large enough such that
\[
\mu\{t \in G : |x_n(t)| > D\} < \delta/4 \quad \text{and} \quad \mu\{t \in G : |x(t)| > D\} < \delta/4.
\]

Hence
\[
\mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon, |x(t)| \leq D, |x_n(t)| \leq D\} > \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}.
\]

If \( t \in G \setminus A \), then \( x(t) \neq a_i, i = 1, 2, \ldots \). Denote \( V_\eta^i = (a_i - \eta, a_i + \eta) \). We have
\[
\lim_{\eta \to 0} \mu\{t \in G \setminus A : x(t) \in V_\eta^i\} = \mu\{t \in G \setminus A : x(t) = a_i\} = 0.
\]

Select \( \eta_i \) such that
\[
\mu\{t \in G \setminus A : x(t) \in V_{\eta_i}^i\} < \frac{\delta}{4 \cdot 2^i}, \quad i = 1, 2, \ldots.
\]

Then
\[
\mu\{t \in G \setminus A : x(t) \in \bigcup_{i=1}^{\infty} V_{\eta_i}^i\} < \frac{\delta}{4}.
\]

Therefore
\[
\mu_G^A = \mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon, |x_n(t)| \leq D, |x(t)| \leq D, \\
x(t) \in R \setminus \bigcup_{i=1}^{\infty} V_{\eta_i}^i\} > \frac{\delta}{4}.
\]

Consider the bounded closed set
\[ F = \left\{(u, v) \in R^2 : |u - v| \geq \varepsilon, |u| \leq D, |v| \leq D, v \in R \bigcup_{i=1}^{\infty} V_i \right\}. \]

It is easy to see that
\[ M\left(\frac{u + v}{2}\right) \leq \frac{M(u) + M(v)}{2}, \]
for \((u, v) \in F\). Since the set \( F \) is compact, there is \( \delta' > 0 \) such that
\[ \max_{(u, v) \in F} \frac{2M\left(\frac{u + v}{2}\right)}{M(u) + M(v)} = 1 - \delta' \]
Hence,
\[ M\left(\frac{u + v}{2}\right) \leq (1 - \delta') \frac{M(u) + M(v)}{2} \]
for any \((u, v) \in F\). Consequently
\[ M\left(\frac{x_n(t) + x(t)}{2}\right) \leq (1 - \delta') \frac{M(x_n(t)) + M(x(t))}{2} \]
for \( \mu\)-a.e. \( t \in G_n \). Therefore
\[
0 \leq \frac{\rho_M(x_n) + \rho_M(x)}{2} - \rho_M\left(\frac{x_n + x}{2}\right) = \int_G \left[ \frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t) + x(t)}{2}\right) \right] dt \\
\geq \int_{G_n} \left[ \frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t) + x(t)}{2}\right) \right] dt \\
\geq \int_{G_n} \delta' \frac{M(x_n(t)) + M(x(t))}{2} dt \geq \frac{\delta'}{2} M\left(\frac{\varepsilon}{2}\right)^\delta > 0
\]
and we get a contradiction. This means that \( x_n(t) \rightarrow^\mu x(t) \) on \( G \setminus A \). By the lower semicontinuity of the modular, we have
\[
\liminf_n \rho_M(x_n|_{G \setminus A}) \geq \rho_M(x|_{G \setminus A}).
\]
Since \( \rho_M(x_n) \rightarrow 1 = \rho_M(x) \), the above inequality implies
\[
\limsup_n \rho_M(x_n|_A) \leq \rho_M(x|_A). \tag{1}
\]
Since $a$ is a left extreme point of $\text{SAI}$ of $M$ but it is not a right extreme point. Then, using the same argumentation as above, we have

$$\liminf_{n} x_{n}(t) \geq x(t)$$

for $\mu$-a.e. $t \in A$. Combining (1), we obtain

$$\lim_{n \to \infty} x_{n}(t) = x(t)$$

for $\mu$-a.e. $t \in A$. Hence $x_{n} \xrightarrow{\mu} x$ on $G$.

Now we prove that

(2) $$\limsup_{\mu \to 0 \atop n} \rho_{M}(x_{n}|_{e}) = 0.$$  

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \in \Sigma$ with $\mu e < \delta$ we get $\rho_{M}(x_{n}|_{e}) < \varepsilon$. Since $x_{n} \xrightarrow{\mu} x$ on $G$, by the Egoroff theorem, there is $e \subset G$ such that $\mu e < \delta$ and $x_{n}(t) \to x(t)$ uniformly on $G \setminus e$. Hence

$$|\rho_{M}(x_{n}|_{G \setminus e}) - \rho_{M}(x|_{G \setminus e})| < \varepsilon$$

for $n \leq n_{0}$. By the convergence $\rho_{M}(x_{n}) \to 1 = \rho_{M}(x)$, we have

$$|\rho_{M}(x_{n}|_{e}) - \rho_{M}(x|_{e})| < \varepsilon$$

for $n$ large enough. Therefore $\rho_{M}(x_{n}|_{e}) < 2\varepsilon$ for $n$ large enough. Hence for any fixed $n$ and any $e \in \Sigma$ with $\mu e$ sufficiently small, we have

$$\rho_{M}(x_{n}|_{e}) < \varepsilon$$

uniformly, i.e. (2) holds true.

By (2) and Lemma 2, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu e < \delta$ implies that

$$\rho_{M}(x|_{e}) < \varepsilon, \quad \rho_{M}(x_{n}|_{e}) < \varepsilon \quad (n = 1, 2, \ldots)$$

and

$$\rho_{N}(y|_{e}) < \varepsilon \quad \text{for} \quad y \in \nabla x.$$  

Since $x_{n} \xrightarrow{\mu} x$, applying again the Egoroff theorem, a set $e \in \Sigma$ can be found such that $\mu e < \delta$ and $x_{n} \to x$ uniformly on $G \setminus e$. Hence,

$$|x_{n}(t) - x(t)| < \varepsilon$$

for $t \in G \setminus e$ and $n$ large enough. Thus

$$<x_{n} - x, y> = \int_{G \setminus e} (x_{n}(t) - x(t)) y(t) dt + \int_{e} (x_{n}(t) - x(t)) y(t) dt \leq$$
\[ \leq \varepsilon \| \mathcal{X}_G \| + \rho_M(\| x_n \|_e \varepsilon) + \rho_M(\| x \|_e \varepsilon) + 2 \rho_N(\| y \|_e \varepsilon) < \varepsilon \| \mathcal{X}_G \| + 4\varepsilon \]

uniformly with respect to \( y \in \nabla_x \), i.e. \( x \) is an U-point.

(II) \( M \in \nabla_2 \) and \( \mu \{ t \in G : x(t) \in (a, b) \} = 0 \) for any SAI \([a, b]\) of \( M \).

Denote
\[ B = \left\{ t \in G : x(t) \in \bigcup_{i=1}^{\infty} \{ b_i \} \right\} \]

First, we will prove that
\[ \lim_{\mu \to 0} \sup_n \rho_M(\| x_n \|_e) = 0. \]

Otherwise, there exist \( e_n \subset G \) with \( \mu e_n \downarrow 0 \) such that \( \rho_M(\| x_n \|_e) \geq \varepsilon \).

Without loss of generality we can assume, \( x_n(t) \geq u_0 \) for \( t \in e_n \). Since \( M \in \nabla_2 \), there exists \( \delta \in (0,1) \) such that for \( u > u_0 \)
\[ M\left( \frac{u}{1+\tau} \right) \leq \frac{1}{1+\tau'}(1-\delta)M(u), \]

where \( \frac{1}{1+\tau'} = 1+\tau \). Hence
\[
1 \leq \rho_M\left( \frac{x_n(t) + x(t)}{2} \right) = \\
= \int_{G \setminus e_n} M\left( \frac{x_n(t) + x(t)}{2} \right) dt + \int_{e_n} M\left( \frac{1+\tau'}{2} \cdot \frac{x_n(t)}{1+\tau'} + \frac{1-\tau'}{2} \cdot x(t) \right) dt \leq \\
\leq \int_{G \setminus e_n} M\left( \frac{x_n(t) + x(t)}{2} \right) dt + \frac{1+\tau'}{2} \int_{e_n} M\left( \frac{x_n(t)}{1+\tau'} \right) dt + \frac{1-\tau'}{2} \int_{e_n} M((1+\tau)x(t)) dt \leq \\
\leq \frac{1}{2} \int_{G \setminus e_n} M(x_n(t)) dt + \frac{1}{2} \int_{e_n} M(x(t)) dt + \frac{1-\delta}{1+\tau'} \frac{1+\tau'}{2} \int_{e_n} M(x_n(t)) dt + \\
+ \rho_M((1+\tau)x|_{e_n}) \leq \\
\leq \frac{\rho_M(x_n)}{2} + \frac{\rho_M(x)}{2} - \frac{\delta \varepsilon}{2} + \rho_M((1+\tau)x|_{e_n}) \to 1 - \frac{\delta \varepsilon}{2}.
\]

This contradiction proves (3).

Analogously as in the proof of (1), we can deduce that \( x_n \xrightarrow{\mu} x \) on \( GB \).

Using (3), we obtain
\[ \lim_{n \to \infty} \rho_M(x_n|_{G|B}) = \rho_M(x|_{G|B}) \]

and consequently
\[ \lim_{n \to \infty} \rho_M(x_n|_B) = \rho_M(x|_B). \]

Since \(b\) is a right extreme point of SAI of \(M\) and not a left extreme point, repeating the same argumentation as above, we get
\[ \lim_{n \to \infty} x_n(t) = x(t), \quad \mu - \text{a.e. on } B. \]

Thus, we obtain that \(x_n(t) \xrightarrow{\mu} x(t)\) on \(G\). Similarly as in the proof of (I), we can verify that \(x\) is an \(U\)-point.

(III) \(M \in \nabla_2\) and \(x(t)\) does not belong to any SAI of \(M\) or \(x(t)\) belongs to some SAI and \(p_+(x(t)) = p(x(t))\).

For every \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any \(e \in \Sigma\) with \(\mu e < \delta\) we have
\[ \rho_M(x|_e) < \varepsilon, \quad \rho_M(x_n|_e) < \varepsilon \quad \text{and} \quad \rho_N(kv|_e) < \varepsilon \]
for each \(v \in \nabla_x\), and \(k \in K(v)\). Denote
\[ E_i = \{ t \in G : x(t) \in [a_i, b_i] \} \quad (i = 1, 2, \ldots) \]
and
\[ E_0 = G \setminus \bigcup_{i=1}^{\infty} E_i. \]

Analogously as in the proof of (I), we can prove \(x_n \xrightarrow{\mu} x\) on \(E_0\). Choose \(F_0 \subset E_0\) such that \(\mu F_0 < \delta\) implies \(x_n(t) \to x(t)\) uniformly on \(E_0 \setminus F_0\), i.e. there exists \(n_0 \in N\) such that \(n > n_0\) implies
\[ |x_n(t) - x(t)| < \varepsilon \]
and
\[ |M(x_n(t)) - M(x(t))| < \varepsilon \]
for \(t \in E_0 \setminus F_0\). Hence, for \(n > n_0\), we have
\[ x_n(t)kv(t) > (x(t) - \varepsilon)kv(t) = M(x(t)) + N(kv(t)) - \varepsilon kv(t) > \]
\[ > M(x_n(t)) - \varepsilon + N(kv(t)) - \varepsilon kv(t) \]
for all \(t \in E_0 \setminus F_0\). Since \(\bigcup_{i=1}^{\infty} \mu E_i \leq \mu G\), there is \(m \in N\) such that \(\mu(\bigcup_{i=m}^{\infty} E_i) < \delta\).

We have
for $u \in [a_i, b_i]$. So there exists $\beta > 0$ such that
\[
up(a_i) > M(u) + N(p(a_i)) - \varepsilon
\]
whenever $u \in [a_i - \beta, b_i + \beta]$ ($i=1, 2, \ldots, m$). Fixing $\beta > 0$, by Lemma 2, we can find $\eta > 0$ such that
\[
\frac{M(u) + M(v)}{2} - M\left(\frac{u + v}{2}\right) < \eta
\]
and for $\nu \in [a_i, b_i]$, imply $u \in [a_i - \beta, b_i + \beta]$ ($i=1, 2, \ldots, m$). Since
\[
f_n(t) = \frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t) + x(t)}{2}\right) \rightarrow 0.
\]
Then
\[
\mu F_n = \mu \{ t \in G : f_n(t) \geq \eta \} < \delta
\]
for $n > n'_0 > n_0$. Hence, if $t \in \bigcup_{i=1}^{m} E_i \setminus F_n$, then $x(t) \in [a_i, b_i]$ and $f_n(t) < \eta$. So
\[
x_n(t) \in [a_i - \beta, b_i + \beta] \quad (i=1, 2, \ldots, m). \quad \text{Furthermore,}
\]
\[
x_n(t) p(a_i) > M(x_n(t)) + N(p(a_i)) - \varepsilon \quad (i=1, 2, \ldots, m).
\]
By the definition of $E_i$, we conclude $x(t) \in [a_i, b_i]$. Hence for $\nu \in \nabla_x$,
\[
k\nu(t) = p(x(t)) = p_-(x(t)) = p(a_i),
\]
i.e.
\[
x_n(t) k\nu(t) > M(x_n(t)) + N(k\nu(t)) - \varepsilon
\]
for $t \in \bigcup_{i=1}^{m} E_i \setminus F_n$. Thus, we conclude that
\[
<x_n, k\nu> = \int_{E_0 \cap F_n} x_n(t) k\nu(t) dt + \int_{\bigcup_{i=1}^{m} E_i \setminus F_n} x_n(t) k\nu(t) dt \geq
\]
\[
\int_{E_0 \cap F_n} (M(x_n(t)) + N(k\nu(t))) dt - o(\varepsilon) =
\]
\[
= \int_{G} (M(x_n(t)) + N(k\nu(t))) dt - o(\varepsilon) = 1 + \rho_M(k\nu) - o(\varepsilon) = k - o(\varepsilon).
\]
This implies $(x_n, \nu) > 1 - o(\varepsilon)$, i.e. $(x_n, \nu) \rightarrow 1$. Hence $x$ is an U-point.

Corollary 2. $L_M$ has local U-property iff
\[
(i) \quad M \in \Delta_2
\]
(ii) for any SAI \([a, b]\) of \(M\), \(p_<(a) = p(a) = p_<(b) = p(b)\),

(iii) \(M \in SC[0, \infty)\) or \(M \in \nabla_2\).

**Proof of necessity.** If \(L_M\) has local \(U\)-property, then it has quasi-\(U\)-property. So (i) and (ii) are satisfied. If (iii) is not true, then \(M \notin \nabla_2\) and there exists a SAI \([a, b]\) of \(M\). Pick \(E \subseteq G\), \(F \subseteq G \setminus E\) and \(c\) large enough such that \(0 < \mu E < \mu G\) and \(M(b)\mu E + M(c)\mu F = 1\). Set \(x = b|_E + c|_F\). Then \(\rho_M(x) = 1\) and \(\mu \{t \in G : x(t) \in (a, b]\} \geq \mu E > 0\).

Hence, \(x \in S(L_M)\) is not \(U\)-point. Therefore \(L_M\) cannot have local \(U\)-property. This contradiction proves (iii).

**Proof of sufficiency.** Using the same argumentation as in the proof of Corollary 1, conditions (i) and (ii) of Theorem 2 hold true. If \(M \in \nabla_2\) or \(M \in SC[0, \infty)\), then \(\mu \{t \in G : |x(t)| \in (a, b]\} = \mu \partial = 0\). Hence \(x\) is an \(U\)-point and consequently \(L_M\) has local \(U\)-property.

**References**


(Department of Mathematics, Harbin University of Science and Technology, Xuefu Road 52, 150080 Harbin, China)

Received on 8.08.1994 and, in revised form, on 10.12.1997.