ON A CONSTRUCTION OF HYPOPARABOLIC POLYNOMIALS
OF TWO SPATIAL VARIABLES AND ITS APPLICATION

ABSTRACT: The aim of the paper is to construct hypoparabolic polynomials of two
spatial variables satisfying an hypoparabolic equation and to apply the theory of those
polynomials for a finding of a solution of a initial-boundary value problem.

KEY WORDS: hypoparabolic polynomials, hypoparabolic equation, initial-boundary
value problem, existence of a solution.

1. INTRODUCTION

In this paper we construct hypoparabolic polynomials of two space variables
satisfying the equation

\[(1) \quad Lu(x,t) = 0, \quad x = (x_1, x_2) \in R^2, \quad t \in R,\]

where

\[L := D_{x_2}^2 - D_{x_1}^2 - D_t.\]

For this purpose we prove two theorems on the forms of \(k\)-hyperbolic polynomials
and we prove a theorem on properties of hypoparabolic functions.

In the paper we also give a theorem on an application of the hypoparabolic
functions for finding of a solution of a initial-boundary value problem. Moreover,
we present six constructions of hyperbolic polynomials in particular cases.

Some methods of parabolic and polyharmonic polynomials were also used in
[1-4].

2. CONSTRUCTIONS OF \(k\)-HYPERBOLIC POLYNOMIALS

Let us consider the homogeneous polynomial \(W_m\), given by the formula

\[(2) \quad W_m(x) = \sum_{j=0}^{m} a_{m-j,j} x_2^{m-j} x_1^j,\]

where \(a_{m-j,j} \quad (j = 0, 1, ..., m)\) are the constant real coefficients.

Let \(H := D_{x_2}^2 - D_{x_1}^2\) and \(H^n := H(H^{n-1}) \quad (n \in N)\). Suppose that the
polynomial \(W_m\) satisfies the conditions
\[ H^n W_m(x) \neq 0, \quad H^{n+1} W_m(x) = 0. \]

It follows, by (3), that \( m = 2n \) or \( m = 2n + 1 \).

We call the polynomials \( W_{2k+r}^i \) (\( i = 0,1,\ldots,2k-1; r = 0,1 \)) satisfying the equations \( H^k W_{2k+r}^i (x) = 0 \), where
\[
W_{2k+r}^i (x) = \sum_{j=0}^{2k+r} a_{2k+r-j,j} x_2^{2k-j} x_1^j
\]
(\( i = 0,1,\ldots,2k-1; r = 0,1 \)), \( k \)-hyperbolic polynomials.

At first, we shall construct the polynomials \( W_0^0, W_j^i \) (\( i = 0,1; j = 1,2,3 \)) and \( W_j^i \) (\( i = 0,1,2,3; j = 4,5 \)).

By \( C_i \) (\( i = 1,2,\ldots,2k; k \in N \)), we denote real constants.

**Construction 1.** There exists one linearly independent homogeneous hyperbolic polynomial \( W_0^0 \), given by the formula \( W_0^0 (x) = 1 \).

Indeed, let \( W_0^0 (x) = a_{0,0} \cdot 1 \), where \( a_{0,0} \neq 0 \). We have that \( HW_0^0 (x) = 0 \). Consequently, the function \( W_0^0 \) is a solution of equation (1). Moreover, this solution is linearly independent.

**Construction 2.** The polynomials \( W_1^0 \) and \( W_1^1 \), given by the formulae \( W_1^0 (x) = x_2 \) and \( W_1^1 (x) = x_1 \), are two linearly independent homogeneous hyperbolic polynomials of the first degree.

Since \( W_1^i (x) = a_{i,0} x_2 + a_{0,1} x_1 \) then \( HW_1^i (x) = 0 \). Applying the Kronecker matrix \( [K_{ij}] \) (\( i,j = 0,1 \)) we obtain that \( W_1^0 (x) = K_{0,0} x_2 \) and \( W_1^1 (x) = K_{1,1} x_1 \).

The polynomials \( W_1^i \) (\( i = 0,1 \)) are linearly independent. Indeed, differentiating the identity \( C_1 x_2 + C_2 x_1 = 0 \) with respect to \( x_2 \), we obtain that \( C_1 = 0 \). Consequently, \( C_2 x_1 = 0 \) and, therefore, \( C_2 = 0 \).

**Construction 3.** The polynomials \( W_2^0 \) and \( W_2^1 \), given by the formulae \( W_2^0 (x) = x_2^2 + x_1^2 \), and \( W_2^1 (x) = x_1 x_2 \), are two linearly independent homogeneous hyperbolic polynomials of the second degree.

Since \( W_2^i (x) = a_{i,0} x_2^2 + a_{i,1} x_2 x_1 + a_{0,2} x_2^2 \) then \( HW_2^i (x) = 2a_{2,0} - 2a_{0,2} = 0 \). Applying the Kronecker matrix, we obtain that
\[ W^0_2(x) = K_{0,0} x_2^2 + K_{0,1} x_2 x_1 + K_{0,2} x_1^2 = x_2^2 + x_1^2 \]

and

\[ W^1_2(x) = K_{1,0} x_2^2 + K_{1,1} x_2 x_1 + K_{1,2} x_1^2 = x_2 x_1. \]

The polynomials \( W^i_2 \) \((i = 0, 1)\) are linearly independent. Indeed, differentiating twice the identity \( C_1 (x_1^3 + x_1^2) + C_2 x_2 x_1 = 0 \), with respect to \( x_2 \), we get \( 2C_1 = 0 \). Hence, we have that \( C_2 = 0 \).

Let

\[ c_{p,q,r,s} := (c_{p,q})^{-1} c_{r,s}, \]

where \( c_{p,q} \) and \( c_{r,s} \) are coefficients.

**Construction 4.** The polynomials \( W^0_3 \) and \( W^1_3 \), given by the formulae

\[ W^0_3(x) = x_2^3 + 3 x_2^2 x_1, \quad \text{and} \quad W^1_3(x) = x_2^2 x_1 + 1/3 x_1^3, \]

are two linearly independent homogeneous hyperbolic polynomials of the third degree.

From the equation \( HW^0_3(x) = 0 \), where

\[ W^0_3(x) = \sum_{j=0}^{3} a_{3-j,j} x_2^{3-j} x_1^j, \]

we obtain the system of the equations

\[ (3!) a_{3,0} - (2!) a_{1,2} = 0 \]

and

\[ (2!) a_{2,1} - (3!) a_{0,3} = 0. \]

Applying (4), we have that \( c_{3,0} = 3! \), \( c_{1,2} = 2! \), \( c_{2,1} = 2! \), \( c_{0,3} = 3! \), \( c_{1,2,3,0} = 3 \) and \( c_{0,3,2,1} = 1/3 \). By Kronecker matrix, we obtain the formulae

\[ W^0_3(x) = \sum_{j=0}^{1} K_{0,j} x_2^{3-j} x_1^j + c_{1,2,3,0} K_{0,0} x_2 x_1^2 + c_{0,3,2,1} K_{0,1} x_1^3 \]

and

\[ W^1_3(x) = \sum_{j=0}^{1} K_{1,j} x_2^{3-j} x_1^j + c_{1,2,3,0} K_{1,0} x_2 x_1^2 + c_{0,3,2,1} K_{1,1} x_1^3. \]

Similarly to the argument given for \( W^i_2 \) \((i = 0, 1)\), differentiating three times, with respect to \( x_2 \), the both sides of the following identity:

\[ C_1 (x_2^3 + 3 x_2 x_1^2) + C_2 (x_2^2 x_1 + 1/3 x_1^3) = 0, \]

we obtain that \( C_1 = C_2 = 0 \).
Construction 5. The polynomials $W^i_4$ ($i = 0, 1, 2, 3$), given by the formulae

\[ W^0_4(x) = x^4_2 - x^4_1, \quad W^1_4(x) = x^2_2 x^2_1 + 1/3 x^4_2, \quad W^2_4(x) = x^3_2 x_1 \quad \text{and} \quad W^3_4(x) = x^3_1 x_2, \]

are four linearly independent homogeneous bihyperbolic polynomials of the fourth degree.

From the equation $H^2 W_4(x) = 0$, where

\[ W_4(x) = \sum_{j=0}^{4} a_{4-j,j} x^{4-j} x^j, \]

we obtain that $(4!)a_{4,0} - 2(2!)(2!)a_{2,2} + (4!)a_{0,4} = 0$. By Kronecker matrix, we have the formulae

\[ W^i_4(x) = \sum_{j=0}^{3} K_{i,j} x^{4-j} x^j + (c_{0,4,4,0} K_{i,0} + c_{0,4,2,2} K_{i,2}) x^i_1 \quad (i = 0, 1, 2, 3), \]

where $c_{0,4,4,0} = -1$ and $c_{0,4,2,2} = 1/3$.

Similarly as in Construction 4, we can prove that $W^i_4$ ($i = 0, 1, 2, 3$) are linearly independent.

Construction 6. The polynomials $W^i_5$ ($i = 0, 1, 2, 3$), given by the formulae

\[ W^0_5(x) = x^5_2 - 5x^4_1 x_2, \quad W^1_5(x) = x^4_2 x_1 - 1/5 x^5_1, \quad W^2_5(x) = x^3_2 x^2_1 + x_2 x^4_1 \quad \text{and} \quad W^3_5(x) = x^2_2 x^3_1 + 1/5 x^5_1, \]

are four linearly independent homogeneous bihyperbolic polynomials of the fifth degree.

From the equation $H^2 W_5(x) = 0$, where

\[ W_5(x) = \sum_{j=0}^{5} a_{5-j,j} x^{5-j} x^j, \]

we obtain the system

\[ a_{5,0} (5!) (0!) - 2(3!)(2!)a_{3,2} + (1!)(4!)a_{1,4} = 0 \]

and

\[ a_{4,1} (4!)(1!) - 2(2!)(3!)a_{2,3} + (0!)(5!)a_{0,5} = 0. \]

Consequently, we have the formulae

\[ W^i_5(x) = \sum_{j=0}^{3} K_{i,j} x^{5-j} x^j + (c_{1,4,5,0} K_{i,0} + c_{1,4,3,2} K_{i,2}) x_2 x^4_1 + \]

\[ + (c_{0,5,4,1} K_{i,1} + c_{0,5,2,3} K_{i,3}) x^5_1 \quad (i = 0, 1, 2, 3), \]

where $c_{1,4,5,0} = -5$, $c_{1,4,3,2} = 1$, $c_{0,5,4,1} = -1/5$ and $c_{0,5,2,3} = 1/5$. 
Similarly, as in Construction 4, we can prove that $W^i_j$ $(i = 0, 1, 2, 3)$ are linearly independent.

**Theorem 1.** If $m = 2k$ then there exist $2k$ linearly independent homogeneous $k$-hyperbolic polynomials $W^i_{2k}$ $(j = 0, 1, \ldots, 2k - 1)$ of degree $2k$, given by the formulae

$$W^{2i}_{2k}(x) = x_2^{2k-2i}x_1^{2i} - c_{0, 2k, 2i, 2i} x_1^{2k} \quad (i = 0, 1, \ldots, k - 1)$$

and

$$W^{2i+1}_{2k}(x) = x_2^{2k-(2i+1)}x_1^{2i+1} \quad (i = 0, 1, \ldots, k - 1),$$

where

$$c_{0, 2k, 2i, 2i} = [(-1)^k (2k)!]^{-1} (2k - 2i)!(2i)!.$$ 

**Proof.** Since $W^i_{2k}(x) = \sum_{j=0}^{2k} a_{2k-j, j} x_2^{2k-j} x_1^j$ and $H^k W^i_{2k}(x) = 0$ then we get that

$$H^k W^i_{2k}(x) = \sum_{j=0}^{2k} a_{2k-j, j} D_{x_2}^{2k} (x_2^{2k-j} x_1^j) +$$

$$- \binom{k}{j}\sum_{j=0}^{2k} a_{2k-j, j} D_{x_2}^{2k-2} D_{x_1}^2 (x_2^{2k-j} x_1^j) + \ldots +$$

$$+ (-1)^p \binom{k}{p} \sum_{j=0}^{2k} a_{2k-j, j} D_{x_2}^{2k-2p} D_{x_1}^{2p} (x_2^{2k-j} x_1^j) + \ldots +$$

$$+ (-1)^{k-1} \binom{k}{k-1} \sum_{j=0}^{2k} a_{2k-j, j} D_{x_1}^k (x_2^{2k-j} x_1^j) +$$

$$+ (-1)^k \binom{k}{k} \sum_{j=0}^{2k} a_{2k-j, j} D_{x_1} (x_2^{2k-j} x_1^j) = 0.$$ 

Calculating in the above equality the suitable derivatives, we obtain

$$\sum_{j=0}^{2k} a_{2k-j, j} (2k-j) \cdots (1-j) x_2^{-j} x_1^j - \binom{k}{j}\sum_{j=0}^{2k} a_{2k-j, j} (2k-j) \cdots x_2^{-j} x_1^j$$

$$\times (3-j) (j-1) x_2^{-j} x_1^{-2} + \ldots + (-1)^p \binom{k}{p} \sum_{j=0}^{2k} a_{2k-j, j} (2k-j) \cdots x_2^{-j} x_1^{-2p} + \ldots +$$

$$\times (2p+1-j) (j-1) \cdots (j-2p+1) x_2^{-p-j} x_1^{-2p} + \ldots +$$
\[
+ (-1)^{k-1} k \sum_{j=0}^{2k} a_{2k-j,j} (2k - j)(2k - j - 1) j(j - 1) \cdots x
\times (j - 2k + 3)x_2^{2k-j-2} x_1^{-(2k-2)} + \\
+ (-1)^k \sum_{j=0}^{2k} a_{2k-j,j} (j-1) \cdots (j - 2k + 1)x_2^{2k-j} x_1^{j-2k} = \\
= \sum_{j=0}^{0} a_{2k-j,j} (2k - j) \cdots (1 - j)x_2^{-j} x_1^{j} + \\
- \binom{k}{1} \sum_{j=0}^{0} a_{2k-j-2,j+2} (2k - j - 2) \cdots (1 - j)(j + 2)(j + 1)x_2^{-j} x_1^{j} + \\
+ \cdots + (-1)^p \binom{k}{p} \sum_{j=0}^{0} a_{2k-j-2p,j+2p} (2k - j - 2p) \cdots x
\times (1 - j)(j + 2p)(j + 2p - 1) \cdots (j + 1)x_2^{-j} x_1^{j} + \cdots + \\
+ (-1)^{k-1} k \sum_{j=0}^{0} a_{2k-j,j+2k-2} (2k - j)(1 - j)(j + 2k - 2)(j + 2k - 3) \cdots x
\times (j + 1)x_2^{-j} x_1^{j} + (-1)^k \sum_{j=0}^{0} a_{2k-j+2k,j+2k} (j + 2k) \times \\
\times (j + 2k - 1) \cdots (j + 1)x_2^{-j} x_1^{j} = 0.
\]

By the last formula, we get

\[
(5) \quad \binom{k}{0} (2k)! a_{2k,0} - \binom{k}{1} (2k-2)! (2)! a_{2k-2,2} + \cdots + \\
+ (-1)^p \binom{k}{p} (2k-2p)! (2p)! a_{2k-2p,2p} + \cdots + \\
+ (-1)^{k-1} \binom{k}{k-1} (2)! (2k-2)! a_{2k-2k-2} + (-1)^k \binom{k}{k} (2k)! a_{0,2k} = 0.
\]

From equation (5), we obtain that

\[
a_{0,2k} = -\sum_{r=0}^{k-1} c_{0,2k,2k-2r,2r} a_{2k-2r,2r},
\]
where
\[ c_{0,2k,2k-2r,2r} = \left( -1 \right)^k (2k)! \left( \begin{array}{c}
k \\
r \end{array} \right) (2k-2r)! (2r)! \]

\((r = 0,1,...,p,...,k-1).\)

The coefficient \(a_{2k-j,j} (j = 0,1,...,2k-1)\) are arbitrary.

Let
\[ W^i_{2k} (x) = \sum_{j=0}^{2k} a^i_{2k-j,j} x_2^{2k-j} x_1^j \quad (i = 0,1,...,2k-1), \]

where
\[ a^i_{2k-j,j} = K_{i,j} \quad (i, j = 0,1,...,2k-1) \]

and
\[ a^i_{0,2k} = -c_{0,2k,2k,0} K_{i,0} - c_{0,2k,2k-2,2} K_{i,2} - c_{0,2k,2k-2p,2p} K_{i,2p} + \]
\[ + ... - c_{0,2k,2k-2,2k-2} K_{i,2k-2} \quad (i = 0,1,...,2k-1). \]

Observe that
\begin{equation}
W^i_{2k} (x) = \sum_{j=0}^{2k-1} K_{i,j} x_2^{2k-j} x_1^j + (-c_{0,2k,2k,0} K_{i,0} - c_{0,2k,2k-2,2} K_{i,2} - ... +
\]
\[ - c_{0,2k,2k-2p,2p} K_{i,2p} - ... - c_{0,2k,2k-2,2k-2} K_{i,2k-2} \right) x_1^{2k} \]
\[ (i = 0,1,...,2k-1). \]

By (6), we obtain that
\[ W^{2i}_{2k} (x) = x_2^{2k-2i} x_1^{2i} - c_{0,2k,2k-2i,2i} x_1^{2k} \quad (i = 0,1,...,k-1) \]

and
\[ W^{2i+1}_{2k} (x) = x_2^{2k-(2i+1)} x_1^{2i+1} \quad (i = 0,1,...,k-1). \]

The polynomials \(W^i_{2k} (i = 0,1,...,2k-1)\) are linearly independent. Indeed, if
\begin{equation}
\sum_{i=0}^{2k-1} C_{i+1} W^i_{2k} (x) = 0 \tag{7}
\end{equation}

then differentiating 2k-times, with respect to \(x_2\), the both sides of equation (7), we obtain that \(C_{i+1} = 0 \quad (i = 0,1,...,2k-1).\)
Theorem 2. If $m = 2k + 1$ then there exist $2k$ linearly independent homogeneous $k$-hyperbolic polynomials $W_{2k}^j$ ($j = 0, 1, \ldots, 2k - 1$) of degree $2k + 1$, given by the formulae

$$W_{2k+1}^{2i}(x) = x_2^{2k+1-2i}x_1^{2i} - c_{1,2k,2k+1,2i} x_2 x_1^{2k}$$

(i = 0, 1, \ldots, k - 1)

and

$$W_{2k+1}^{2i+1}(x) = x_2^{2k+1-(2i+1)}x_1^{2i+1} - c_{0,2k+1,2k-2i,2i+1} x_1^{2k+1}$$

(i = 0, 1, \ldots, k - 1),

where

$$c_{1,2k,2k-2i,2i} = [(-1)^k (2k)!]^{-1} (2k - 2i + 1)! (2i)!$$

(i = 0, 1, \ldots, k - 1)

and

$$c_{0,2k+1,2k-2i,2i+1} = [(-1)^k (2k + 1)!]^{-1} (2k - 2i)! (2i + 1)!$$

(i = 0, 1, \ldots, k - 1)

Proof. Since $W_{2k+1}(x) = \sum_{j=0}^{2k+1} a_{2k+1-j,j} x_2^{2k+1-j} x_1^j$ satisfies the equation

$H^k W_{2k+1}(x) = 0$ then we have

$$H^k W_{2k+1}(x) = \sum_{j=0}^{2k+1} a_{2k+1-j,j} (D_{x_2}^2 - D_{x_1}^2)^k x_2^{2k+1-j} x_1^j =$$

$$= \sum_{j=0}^{2k+1} a_{2k+1-j,j} D_{x_2}^{2k} (x_2^{2k+1-j} x_1^j) +$$

$$- \binom{k}{1} \sum_{j=0}^{2k+1} a_{2k+1-j,j} D_{x_2}^{2k-2} D_{x_1}^2 (x_2^{2k+1-j} x_1^j) +$$

$$+ \vdots + (-1)^p \binom{k}{p} \sum_{j=0}^{2k+1} a_{2k+1-j,j} D_{x_2}^{2k-2p} D_{x_1}^{2p} (x_2^{2k+1-j} x_1^j) + \ldots +$$

$$+ (-1)^{k-1} \binom{k}{k-1} \sum_{j=0}^{2k+1} a_{2k+1-j,j} D_{x_2}^{2k-2} D_{x_1}^2 (x_2^{2k+1-j} x_1^j) +$$

$$+ (-1)^k \binom{k}{k} \sum_{j=0}^{2k+1} a_{2k+1-j,j} D_{x_1}^{2k} (x_2^{2k+1-j} x_1^j) =$$

$$= \sum_{j=0}^{2k+1} a_{2k+1-j,j} (2k + 1 - j) \ldots (2 - j) (x_2^{1-j} x_1^j) +$$
\[-\left( \binom{k}{1} \sum_{j=0}^{2k+1} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \cdot (4-j) j(j-1) \times \right.
\times (x_2^{2-j} x_1^{j-2}) + \ldots + (-1)^p \left( \binom{k}{p} \sum_{j=0}^{2k+1} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \times \right.
\times (2p+2-j) j(j-1) \cdot \ldots \cdot (j-2p+1)x_2^{2p+1-j} x_1^{j-2p} + \ldots +
\left. +(-1)^{k-1} \left( \binom{k}{k-1} \sum_{j=0}^{2k+1} a_{2k+1-j,j} (2k+1-j)(2k-j)j \times \right.
\times (j-1) \cdot \ldots \cdot (j-2k+3)x_2^{2k-1-j} x_1^{j-(2k-2)} + \right.
\left. +(-1)^{k-1} \left( \binom{k}{k} \sum_{j=0}^{2k+1} a_{2k+1-j,j} j(j-1) \cdot \ldots \cdot (j-2k+1)x_2^{2k+1-j} x_1^{j-2k} = \right.
= \sum_{j=0}^{1} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \cdot (2-j)x_2^{1-j} x_1^{j} + \right.
\left. - \left( \binom{k}{1} \sum_{j=2}^{3} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \cdot (4-j) j(j-1)(x_2^{3-j} x_1^{j-2}) + \right.
\right.
\left. + \ldots + (-1)^p \left( \binom{k}{p} \sum_{j=2}^{2p+1} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \times \right.
\times (2p+2-j) j(j-1) \cdot \ldots \cdot (j-2p+1)x_2^{2p+1-j} x_1^{j-2p} + \ldots +
\left. +(-1)^{k-1} \left( \binom{k}{k-1} \sum_{j=2k-2}^{2k} a_{2k+1-j,j} (2k+1-j)(2k-j)j \times \right.
\times (j-1) \cdot \ldots \cdot (j-2k+3)x_2^{2k-1-j} x_1^{j-(2k-2)} + \right.
\right.
\left. +(-1)^{k} \left( \binom{k}{k} \sum_{j=2k}^{2k+1} a_{2k+1-j,j} j(j-1) \cdot \ldots \cdot (j-2k+1)x_2^{2k+1-j} x_1^{j-2k} = \right.
= \sum_{j=0}^{1} a_{2k+1-j,j} (2k+1-j) \cdot \ldots \cdot (2-j)x_2^{1-j} x_1^{j} + \right.
\left. - \left( \binom{k}{1} \sum_{j=0}^{1} a_{2k-j-1,j+2} (2k-1-j) \cdot \ldots \cdot (2-j)(j+2)(j+1)x_2^{1-j} x_1^{j} + \right.
\right.
\left. + \ldots + (-1)^p \left( \binom{k}{p} \sum_{j=0}^{1} a_{2k+1-(j+2p),j+2p} (2k+1-j-2p) \cdot \ldots \times \right. \]
\[ \times (2 - j)(j + 2p)(j + 2p - 1) \cdots (j + 1) x_2^{1 - j} x_1^j + \cdots + \]

\[ + (-1)^{k-1} \binom{k}{k-1} \sum_{j=0}^{1} a_{3-j,j+2k-2} (3 - j)(2 - j)(j + 2k - 2) \times \]

\[ \times (j + 2k - 3) \cdots (j + 1) x_2^{1-j} x_1^j + \]

\[ + (-1)^k \binom{k}{k} \sum_{j=0}^{1} a_{1-j,j+2k} (j + 2k)(j + 2k - 1) \cdots (j + 1) x_2^{1-j} x_1^j = \]

\[ = [a_{2k+1,0} (2k + 1)! \binom{k}{1} a_{2k-1,2} (2k - 1)! (2)! + \cdots + \]

\[ + (-1)^{p} \binom{k}{p} a_{2k+1-2p,2p} (2k + 1 - 2p)! (2p)! + \cdots + \]

\[ + (-1)^{k-1} \binom{k}{k-1} a_{3,2k-2} (2k - 2)! (3)! + (-1)^k \binom{k}{k} a_{1,2k} (2k)! ] x_2 + \]

\[ + [a_{2k,1} (2k)! \binom{k}{1} a_{2k-2,3} (2k - 2)! (3)! + \cdots + \]

\[ + (-1)^{p} \binom{k}{p} a_{2k-2p,1+2p} (2k - 2p)! (2p + 1)! + \cdots + \]

\[ + (-1)^{k-1} \binom{k}{k-1} a_{2,2k-1} (2)! (2k - 1) + \]

\[ + (-1)^k \binom{k}{k} a_{0,1+2k} (2k + 1)! ] x_1 = 0. \]

Hence, we get the system of two equations

\[ a_{2k+1,0} (2k + 1)! + a_{2k-1,2} (-k)(2k - 1)! (2)! + \cdots + \]

\[ + a_{2k+1-2p,2p} (-1)^{2p} \binom{k}{p} (2k + 1 - 2p)! (2p)! + \cdots + \]

\[ + a_{3,2k-2} (-1)^{k-1} \binom{k}{k-1} (3)! (2k - 2)! + a_{1,2k} (-1)^{k} \binom{k}{k} (2k)! = 0, \]
\[ a_{2k,1}(2k)! + a_{2k-2,3}(-k)(2k - 2)! (3)! + \ldots + \]
\[ + a_{2k-2p,2p+1}(-1)^p \binom{k}{p}(2k - 2p)! (2p + 1)! + \ldots + \]
\[ + a_{2,2k-1}(-1)^{k-1} \binom{k}{k-1}(2)! (2k - 1)! + \]
\[ + a_{0,2k+1}(-1)^k \frac{k}{k}(2k + 1)! = 0. \]

By (8), we obtain the formulae
\[ a_{1,2k} = -a_{2k+1,0} c_{1,2k,2k+1,0} - a_{2k-1,2} c_{1,2k,2k-1,2} + \]
\[ - \ldots - a_{2k+1-2p,2p} c_{1,2k,2k+1-2p,2p} - \ldots - a_{3,2k-2} c_{1,2k,3,2k-2} \]
and
\[ a_{0,2k+1} = -a_{2k+1,0} c_{0,2k+1,2k,1} - a_{2k-2,3} c_{0,2k+1,2k-2,3} + \]
\[ - \ldots - a_{2k-2p,2p+1} c_{0,2k+1,2k-2p,2p+1} - \ldots - a_{2,2k-1} c_{0,2k+1,2,2k-1}, \]
where
\[ c_{1,2k,2k+1-2r,2r} = (\frac{(-1)^k}{(2k)!})^{-1} (2k + 1 - 2r)! (2r)! \]
and
\[ c_{0,2k+1,2k-2r,2r+1} = (\frac{(-1)^k}{(2k+1)!})^{-1} (2k - 2r)! (2r + 1)! \]
\[ (r = 0,1,\ldots,p,\ldots,k-1). \]

The coefficients \( a_{2k+1-2r,2r}, a_{2k-2r,2r+1} \) \((r = 0,1,\ldots,p,\ldots,k-1)\) are arbitrary.

Let
\[ W_{2k+1}^j(x) = \sum_{j=0}^{2k+1} a_{2k+1-j, j}^j x_2^{2k+1-j} x_1^j, \]
where
\[ a_{2k+1-j,j}^j = K_{i,j} \quad (i,j = 0,1,\ldots,2k-1), \]
\[ a_{1,2k}^i = -c_{1,2k,2k+1,0} K_{i,0} - c_{1,2k,2k-1,2} K_{i,2} + \]
\[ - \ldots - c_{1,2k,2k+1-2p,2p} K_{i,2p} - \ldots - c_{1,2k,3,2k-2} K_{i,2k-2} \]
\[ (i = 0,1,\ldots,2k-1) \]
and
\[ a_{0,2k+1}^i = -c_{0,2k+1,2k,1} K_{i,1} - c_{0,2k+1,2k-2,3} K_{i,3} + \]
\[ - \ldots - c_{0,2k+1,2k-2} r_{2k+1} K_{i,2k-2} - c_{0,2k+1,2k-1} K_{i,2k-1} \]

\[ = - \ldots - c_{0,2k+1,2k-2} p_{2k+1} K_{i,2k-2} - c_{0,2k+1,2k-1} K_{i,2k-1} \]

\[ (i = 0, \ldots, 2k - 1). \]

Observe that

\[ W_{2k+1}^i (x) = \sum_{j=0}^{2k-1} K_{i,j} x_2^{2k+1-j} x_1^j - (c_{i,2k,2k+1,0} K_{i,0} + \]

\[ + c_{i,2k,2k-1,2} K_{i,2} \ldots + c_{i,2k,2k-2,2,k+1,2} K_{i,2} + \]

\[ + c_{i,2k,3,2k-2} K_{i,2k-2} x_2 x_1^{2k} - (c_{i,2k+1,2k,1} K_{i,1} + c_{0,2k+1,2k-2,3} K_{i,3} + \ldots + \]

\[ + c_{0,2k+1,2k-2,2,k+1,2} K_{i,2k-1} x_1^{2k+1} \]

\[ (i = 0, \ldots, 2k - 1). \]

By the above formula, we obtain that

\[ W_{2k+1}^{2i} = x_2^{2k+1-2i} x_1^{2i} - (c_{i,2k,2k-2i+1,2i} x_2 x_1^{2k} \]

\[ (i = 0, \ldots, k - 1) \]

and

\[ W_{2k+1}^{2i+1} = x_2^{2k+1-(2i+1)} x_1^{2i+1} - c_{0,2k+1,2k-2i+1,2i+1} x_1^{2k+1} \]

\[ (i = 0, \ldots, k - 1). \]

The polynomials \( W_{2k+1}^i \) \((i = 0, \ldots, k - 1)\) are linearly independent. Indeed, let

\[ \sum_{i=0}^{2k-1} C_{i+1} W_{2k+1}^i (x) = 0. \]

Differentiating \( 2k + 1 \) times, with respect to \( x_2 \), the both sides of equation (9), we obtain that \( C_{i+1} = 0 \) \((i = 0, \ldots, 2k - 1)\).

3. The Hypoparabolic Polynomials

Theorem 3. If the polynomial \( W_m \), defined by formula (2), satisfies condition (3) then the hypoparabolic function \( S_n \), given by the formula

\[ S_n (x,t) = \sum_{j=0}^{n} t^j j! H^j W_m (x), \]

satisfies equation (1).
Proof. Observe that
\[ HS_n(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} H^{j+1} W_m(x) = \sum_{j=0}^{n-1} \frac{t^j}{j!} H^{j+1} W_m(x) + \sum_{j=0}^{n} \frac{t^{j-1}}{(j-1)!} H^j W_m(x) \]
and
\[ D_x S_n(x,t) = \sum_{j=1}^{n} \frac{t^{j-1}}{(j-1)!} H^j W_m(x). \]
Consequently, we obtain (1).

Therefore, the proof is complete.

Remark. The hypoparabolic polynomials are of the forms:
\[ S_{2k}^p(x,t) = W_{2k}^p(x) + t H W_{2k}^p(x) + \frac{t^2}{2} H^2 W_{2k}^p(x) + \ldots + \frac{t^{2k-2}}{(2k-2)!} H^{2k-2} W_{2k}^p(x) \quad (p = 0,1,\ldots,2k-1) \]
and
\[ S_{2k+1}^p(x,t) = W_{2k+1}^p(x) + t H W_{2k+1}^p(x) + \frac{t^2}{2} W_{2k+1}^p(x) + \ldots + \frac{t^{2k-2}}{(2k-2)!} H^{2k-2} W_{2k+1}^p(x) \quad (p = 0,1,\ldots,2k-1). \]

4. AN EXAMPLE OF A SOLUTION TO A INITIAL-Boundary VALUE PROBLEM

Let us consider the problem
\[ (D_{x_2}^2 - D_{x_1}^2 - D_t) u(x,t) = (ax_1 + bx_2 + c)V(t), \quad (x,t) \in D, \]
where
\[ D = \{(x,t) : x = (x_1,x_2), \quad x \in [0,1]^2, \quad t \in [0,T]\}, \]
\[ u(x,0) = S_n(x,0), \quad x \in [0,1]^2, \]
and
\[ u(0,x_2,t) = S_n(0,x_2,t) - (bx_2 + c) \int_0^t V(s) ds, \]
\[ x_2 \in [0,1], \quad t \in [0,T], \]
(14) \[ u(x_1,0,t) = S_n(x_1,0,t) - (ax_1 + c)\int_0^t V(s)ds, \]

\[ x_1 \in [0,1], \quad t \in [0,T], \]

where \( V \) is a polynomial of the variable \( t \), \( S_n \) is given by formula (10) and \( a, b, c \) are constant coefficients.

It is possible to prove the following:

**Theorem 4.** The function \( u \) given by the formula

\[ u(x,t) := S_n(x,t) - (ax_1 + bx_2 + c)\int_0^t V(s)ds, \quad (x,t) \in D, \]

is a solution of problem (11)-(14).

REFERENCES


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