THE VORONOVSKAYA TYPE THEOREM FOR THE BASKAKOV-KANTOROVICH OPERATORS

ABSTRACT: In this paper we give the Voronovskaya type theorem for the Baskakov-Kantorovich operators in the polynomial weighted spaces.

The Baskakov operators were studied in [1]. The Voronovskaya type theorem for some operators are given in [2], [3].

KEY WORDS: Voronovskaya theorem, Baskakov-Kantorovich operator, weighted space.

1. PRELIMINARY

We take the notation like in M. Becker’s paper [1], i.e.: \( N := \{1, 2, \ldots, \} \), \( N_0 := N \cup \{0\} \), \( R_0 := [0, +\infty) \) and let for a fixed \( p \in N_0 \) and for all \( x \in R_0 \)

\[(1) \quad w_0(x) = 1, \quad w_p(x) = \frac{1}{1 + x^p} \quad \text{if} \quad p \geq 1.\]

Let \( C_p = C_p(R_0) \) being the set of all real - valued functions continuous on \( R_0 \), for which \( w_p(\cdot)f(\cdot) \) is uniformly continuous and bounded on \( R_0 \). The norm in \( C_p \) is defined by the formula

\[(2) \quad ||f||_{C_p} := \sup_{x \in R_0} w_p(x)|f(x)|.\]

Let \( m \in N, \ p \in N_0 \), being fixed numbers. Denote by \( C^m_p \), the set of \( f \in C_p \), which \( f^{(k)} \), \( k = 0, 1, \ldots, m \), belong to \( C_p \).

In paper [1] were studied the Baskakov operators for functions \( f \in C_p \)

\[(3) \quad V_n(f; x) = \sum_{k=0}^{\infty} b_{n,k}(x)f\left(\frac{k}{n}\right), \quad x \in R_0, \ n \in N,\]

where

\[b_{n,k}(x) = \binom{n + k - 1}{k} x^k (1 + x)^{-n-k} \quad \text{for} \quad k \in N_0.\]

Now we introduce the Baskakov-Kantorovich operators \( T_n \) for \( f \in C_p \), \( p \in N_0 \),
(4) \[ T_n(f; x) = \sum_{k=0}^{\infty} b_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt, \quad x \in R_0, \ n \in N. \]

From (4) it follows that

(5) \[ T_n(1; x) = 1 \quad \text{for} \quad x \in R_0, \ n \in N. \]

The further properties \( T_n \) we shall give in Section 2. The Voronovskaya type theorem will be given in Section 3.

2. LEMMAS

In this part we will need Becker's results of the Baskakov operators \( V_n \) defined by the formula (3).

Lemma 1 ([1]). Let \( p \in N_0 \) be a fixed number. Then there exists a positive constant \( M_3(p) \) depending only on \( p \) such that for all \( n \in N \) and \( x \in R_0 \) we have

(6) \[ w_p(x)V_n\left(\frac{(t-x)^2}{w_p(x)}; x\right) \leq M_3(p)\frac{x(1+x)}{n}. \]

Now we shall give some properties of the operators \( T_n \).

Lemma 2. For all \( n \in N \) and \( x \in R_0 \) we have

\[ T_n(t-x; x) = \frac{1}{2n}, \]

\[ T_n((t-x)^2; x) = \frac{x^2}{n} + \frac{x}{n} + \frac{1}{3n^2}, \]

(7) \[ T_n((t-x)^4; x) = 3 \frac{x^4}{n^2} + 6 \frac{x^4}{n^3} + 6 \frac{x^3}{n^2} + 16 \frac{x^3}{n^3} + 3 \frac{x^2}{n^2} + 15 \frac{x^2}{n^3} + 5 \frac{x}{n^3} + \frac{1}{5n^4}. \]

Proof. For example we shall prove (7). From the linearity of the operator \( T_n \) we get

\[ T_n((t-x)^4; x) = T_n(t^4; x) + 4xT_n(t^3; x) + 6x^2T_n(t^2; x) - 4x^3T_n(t; x) + x^4T_n(1; x). \]

From (5) we know, that \( T_n(1; x) = 1 \). Using the definition of the operators \( T_n \) we can calculate, that for \( x \in R_0, \ n \in N, \) we have
\[ T_n(t;x) = x + \frac{1}{2n^2}, \]
\[ T_n(t^2;x) = \frac{(n+1)}{n} x^2 + 2 \frac{1}{n} x + \frac{1}{3n^2}, \]
\[ T_n(t^3;x) = \frac{(n+1)(n+2)}{n^2} x^3 + \frac{9}{2} \frac{(n+1)}{n^2} x^2 + \frac{7}{2} \frac{1}{n^2} + \frac{1}{4n^3}, \]
\[ T_n(t^4;x) = \frac{(n+1)(n+2)(n+3)}{n^3} x^4 + \frac{8(n+1)(n+2)}{n^3} x^3 + \]
\[ + 15 \frac{(n+1)}{n^3} x^2 + 6 \frac{1}{n^2} + \frac{1}{5n^4}, \]

which yield that

\[ T_n((t-x)^4;x) = 3 \frac{x^4}{n^2} + 6 \frac{x^4}{n^3} + 6 \frac{x^3}{n^2} + 16 \frac{x^3}{n^3} + 3 \frac{x^2}{n^2} + 15 \frac{x^2}{n^3} + 5 \frac{x}{n^3} + \frac{1}{5n^4} \]

for \( x \in R_0, \ n \in N. \)

Using the mathematical induction we can prove an analogous formula given in [1] for \( V_n(t^q;x), \ q \in N. \)

Lemma 3. For \( q \in N_0, \ n \in N, \ x \in R_0 \) there exist the positive coefficients \( \xi_{j,n,q}, \ 0 \leq j \leq q, \) depending only on \( q, \ j, \ n \) and bounded for \( n \) and \( x \) such that

\[ T_n(t^q;x) = \sum_{j=0}^{q} \xi_{j,n,q} \ x^j \ n^{j-q}, \]

(8)

where \( 1 < \xi_{q,n,q} \leq q!. \)

Now we shall prove

Lemma 4. For every fixed \( p \in N_0 \) there exists a positive constant \( M_1(p) \) depending only on \( p \) such that for all \( n \in N \) we have

\[ \left\| T_n\left( \frac{1}{w_p(t)} \right) \right\|_{C_p} \leq M_1(p). \]

(9)

Proof. By (1), (2) and (5) the inequality (9) is obvious if \( p = 0. \) Fix \( p \in N. \) From (1), (5) and the linearity of the operator \( T_n \) we get for \( x \in R_0 \) and \( n \in N \)
\[
\begin{align*}
w_p(x)T_n \left( \frac{1}{w_p(t)} ; x \right) &= w_p(x)T_n (1 + t^p ; x) \\
&\leq w_p(x) \left( 1 + p!x^p + \sum_{j=0}^{p-1} \xi_{j,n,p} x^j n^{j-p} \right) \\
&\leq p! + \sum_{j=0}^{p-1} \xi_{j,n,p} \frac{x^j}{1 + x^p} n^{j-p}.
\end{align*}
\]

But \(0 \leq x^j/(1 + x^p) \leq 1\) for \(x \in R_0\) and \(1 \leq j \leq p - 1\). Hence

\[
\begin{align*}
w_p(x)T_n \left( \frac{1}{w_p(t)} ; x \right) &\leq p! + \frac{1}{n} \sum_{j=0}^{p-1} \xi_{j,n,p} n^{j-(p-1)} \\
&\leq p! + \frac{1}{n} \sum_{j=0}^{p-1} \xi_{j,n,p} \leq M_1(p) = \text{const.}
\end{align*}
\]

for all \(x \in R_0\) and \(n \in N\), which implies

\[
\left\| T_n \left( \frac{1}{w_p(t)} ; \right) \right\|_{C_p} \leq M_1(p) \quad \text{for all } n \in N.
\]

Thus the proof of (9) is completed.

Applying Lemma 4 we shall prove

**Lemma 5.** For every fixed \(p \in N_0\) there exists a positive constant \(M_1(p)\) depending only on \(p\) such that any \(f \in C_p\)

\[
\|T_n(f;\cdot)\|_{C_p} \leq M_1(p) \|f\|_{C_p}, \quad n \in N,
\]

which proves that \(T_n, n \in N,\) is a linear positive operator from the space \(C_p\) into \(C_p\).

**Proof.** From (2) we have

\[
\|T_n(f;\cdot)\|_{C_p} = \sup_{x \in R_0} w_p(x) |T_n(f; x)|.
\]

But from (4) we get
\[
\begin{align*}
  w_p(x) |T_n(f; x)| &\leq w_p(x) \sum_{k=0}^{\infty} b_{n,k}(x)x^k \left( \int_{k/n}^{(k+1)/n} |f(t)| w_p(t) \frac{1}{w_p(t)} dt \right) \\
  &\leq \|f\|_{C_p} w_p(x) T_n \left( \frac{1}{w_p(t)} ; x \right) \leq \|f\|_{C_p} \left\| T_n \left( \frac{1}{w_p(t)} ; \cdot \right) \right\|_{C_p}.
\end{align*}
\]

Using Lemma 4, we immediately obtain (10).

**Lemma 6.** For some \( x_0 \in R_0 \) there exists a positive constant \( M_2(x_0) \), depending only on \( x_0 \), such that for all \( n \in N \) we have

\[
T_n((t - x_0)^x; x_0) \leq M_2(x_0) n^{-2}.
\]

**Proof.** Applying (7) we get

\[
T_n((t - x_0)^x; x_0) = 3 \frac{x_0^4}{n^2} + 6 \frac{x_0^4}{n^2} + 6 \frac{x_0^3}{n^2} + 16 \frac{x_0^3}{n^2} + 3 \frac{x_0^2}{n^2} + 15 \frac{x_0^2}{n^2} + 4 \frac{x_0}{n^2} + \frac{1}{5n^4} \leq
\]

\[
(9x_0^4 + 22x_0^3 + 18x_0^2 + 5x_0 + 0,2) n^{-2} = M_2(x_0) n^{-2}.
\]

for every \( n \in N \). Hence we have (11).

**Lemma 7.** For every \( x_0 \in R_0 \) holds

\[
\lim_{n \to \infty} nT_n(t - x_0; x_0) = \frac{1}{2},
\]

\[
\lim_{n \to \infty} nT_n((t - x_0)^2; x_0) = x_0(1 + x_0).
\]

**Proof.** From Lemma 2 we know that

\[
T_n(t - x_0; x_0) = \frac{1}{2n^2},
\]

\[
T_n((t - x_0)^2; x_0) = \frac{x_0^2}{n} + \frac{x_0}{n} + \frac{1}{3n^2},
\]

for all \( n \in N \) and \( x_0 \in R_0 \), which immediately yield (12) and (13).

**Lemma 8.** Let \( x_0 \in R_0 \) be a fixed point and let \( g(\cdot ; x_0) \) be a given function belonging to \( C_p \) with some \( p \in N_0 \) and such that

\[
\lim_{t \to x_0} g(t; x_0) = 0.
\]
Then
\begin{equation}
\lim_{t \to x_0} T_n \left( g(t; x_0); x_0 \right) = 0.
\end{equation}

Proof. Choose \( \varepsilon > 0 \) and a constant \( M_1(p) \) like in Lemma 4. Then from (14) and properties of \( g(\cdot; x_0) \) there exist the positive constants \( \delta = \delta(\varepsilon, M_1) \) and \( M_4 \) such that
\begin{align}
&\quad w_p(t)|g(t; x_0)| < \frac{\varepsilon}{2M_1} \quad \text{for} \quad |t - x_0| < \delta, \\
&\quad w_p(t)|g(t; x_0)| < \frac{\varepsilon}{2M_1} \quad \text{for all} \quad t \in R_0.
\end{align}

Notation by \( Q_{n,1} := \{ k \in N_0 : |k/n - x_0| < \delta \} \) and \( Q_{n,2} := \{ k \in N_0 : |k/n - x_0| \geq \delta \} \), then for all \( n \in N \) we have
\begin{align*}
w_p(x_0)T_n \left( g(t; x_0); x_0 \right) &\leq w_p(x_0) \sum_{k=0}^{\infty} b_{n,k}(x_0)n \int_{k/n}^{(k+1)/n} g(t; x_0)dt = \\
&= w_p(x_0) \sum_{k \in Q_{n,1}} b_{n,k}(x_0)n \int_{k/n}^{(k+1)/n} g(t; x_0)dt + \\
&\quad + w_p(x_0) \sum_{k \in Q_{n,2}} b_{n,k}(x_0)n \int_{k/n}^{(k+1)/n} g(t; x_0)dt \equiv \Sigma_1 + \Sigma_2.
\end{align*}

From (16) and Lemma 4 we get
\begin{align*}
\Sigma_1 &\leq \frac{\varepsilon}{2M_1} w_p(x_0) \sum_{k=0}^{\infty} b_{n,k}(x_0)n \int_{k/n}^{(k+1)/n} \frac{1}{w_p(t)}dt \leq \\
&\leq \frac{\varepsilon}{2M_1} w_p(x_0) T_n \left( \frac{1}{w_p(t)}; x_0 \right) \leq \frac{\varepsilon}{2M_1} M_1 = \frac{\varepsilon}{2}.
\end{align*}

Moreover by (17)
\begin{align*}
\Sigma_2 &\leq M_4 w_p(x_0) \sum_{k \in Q_{n,2}} b_{n,k}(x_0)n \int_{k/n}^{(k+1)/n} \frac{1}{w_p(t)}dt.
\end{align*}

But
\begin{align*}
\int_{k/n}^{(k+1)/n} \frac{1}{w_p(t)}dt \leq \frac{1}{n w_p((k+1)/n)} \leq 2^{p+1} \frac{1}{n w_p(k/n)}.
\end{align*}
Thus
\[ \sum_2 \leq M_4 2^{p+1} w_p(x_0) \sum_{k \in \mathbb{Q}_{\geq 2}} b_{n,k}(x_0) \frac{1}{w_p(k/n)}. \]
If \( |k/n - x_0| \geq \delta \) then \( 1 \leq \delta^{-2}(k/n - x_0)^2 \). So
\[ \sum_2 \leq M_4 2^{p+1} \delta^{-2} w_p(x_0) \sum_{k \in \mathbb{Q}_{\geq 2}} b_{n,k}(x_0) \left( \frac{k}{n} - x_0 \right)^2 \frac{1}{w_p(k/n)} \leq \]
\[ \leq M_4 2^{p+1} \delta^{-2} w_p(x_0) \sum_{k=0}^{\infty} b_{n,k}(x_0) \left( \frac{k}{n} - x_0 \right)^2 \frac{1}{w_p(k/n)} \leq \]
\[ \leq M_4 2^{p+1} \delta^{-2} w_p(x_0) V_n \left( \frac{(t-x_0)^2}{w_p(t)} ; x_0 \right). \]
Using Lemma 1 we get
\[ \sum_2 \leq M_4 M_5(p) 2^{p+1} \frac{x_0(1 + x_0)}{n \delta^2} \equiv M_5(p) \frac{x_0(1 + x_0)}{n \delta^2}. \]
For the fixed positive numbers \( \varepsilon, \delta, M_4 \) and \( x_0 \geq 0 \) there exists natural number \( n_1 \), depending only on \( \varepsilon, \delta, M_5 \) such that for all \( n > n_1 \) we have
\[ M_5(p) \frac{x_0(1 + x_0)}{n \delta^2} < \frac{\varepsilon}{2}. \]
Then \( \sum_2 < \varepsilon/2 \) for all \( n > n_1 \).
Consequently,
\[ w_p(x_0) |T_n(g(t;x_0);x_0)| < \varepsilon \quad \text{for all} \quad n > n_1, \]
which denotes
\[ \lim_{n \to \infty} w_p(x_0) T_n(g(t;x_0);x_0) = 0. \]
From this and by the definition (1) we get (15).

3. THE VORONOVSKAYA TYPE THEOREM

Now we shall prove the main theorem

**Theorem.** Let \( f \in C^2_p \) with some \( p \in N_0 \). Then for every \( x \in R_0 \) we have
\begin{equation}
\lim_{n \to \infty} n \{ T_n(f(x)) - f(x) \} = \frac{1}{2} f''(x) + \frac{x(1+x)}{2} f'''(x) .
\end{equation}

Proof. We take a fixed point $x_0 \in R_0$. By the Taylor formula for $f \in C^2$ and 
t \in R_0$ we have 
\begin{equation}
f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2} f''(x_0)(t-x_0)^2 + \varphi(t;x_0)(t-x_0)^2 ,
\end{equation}
where $\varphi(\cdot;x_0) \in C^1$ and $\lim_{t \to x_0} \varphi(t;x_0) = 0$. From this we derive 
\begin{equation}
\begin{split}
T_n(f(t);x_0) &= f(x_0) T_n(1;x_0) + f'(x_0) T_n(t-x_0;x_0) + \\
&\quad + \frac{1}{2} f''(x_0) T_n((t-x_0)^2;x_0) + T_n(\varphi(t;x_0)(t-x_0)^2;x_0)
\end{split}
\end{equation}
for every $n \in N$. By Lemma 7 we get from (19) 
\begin{equation}
\begin{split}
\lim_{n \to \infty} n \{ T_n(f(t);x_0) - f(x_0) \} &= \frac{1}{2} f''(x_0) + \frac{x_0(x_0+1)}{2} f'''(x_0) + \\
&\quad + \lim_{n \to \infty} T_n(\varphi(t;x_0)(t-x_0)^2;x_0).
\end{split}
\end{equation}
We shall prove that 
\begin{equation}
\lim_{n \to \infty} T_n(\varphi(t;x_0)(t-x_0)^2;x_0) = 0.
\end{equation}
Using the Hölder inequality we have 
\begin{equation}
\left| T_n(\varphi(t;x_0)(t-x_0)^2;x_0) \right| \leq \left\{ T_n(\varphi^2(t;x_0);x_0) \right\}^{1/2} \left\{ T_n((t-x_0)^4;x_0) \right\}^{1/2}.
\end{equation}
But by Lemma 6 we have 
\begin{equation}
T_n((t-x_0)^4;x_0) \leq M_2(x_0) n^{-2} \quad \text{for all } n \geq 1.
\end{equation}
Further the properties of function $\varphi(\cdot;x_0)$ let write that $\psi(t;x_0) = \varphi^2(t;x_0)$ 
belongs to $C^2$ and $\lim_{t \to x_0} \psi(t;x_0) = 0$. Hence we get by Lemma 8 
\begin{equation}
\lim_{n \to \infty} T_n(\psi(t;x_0);x_0) = \lim_{n \to \infty} T_n(\varphi^2(t;x_0);x_0) = 0.
\end{equation}
Combining (22) – (24) we obtain (21), which using to (20) we get the desired 
assertion (18).

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REFERENCES


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