SOME REMARKS ABOUT A GENERALIZED METHOD OF SUMMABILITY OF SERIES

ABSTRACT: In this paper we investigate certain methods of summability of series. We obtain a connection between those methods.

KEY WORDS: singular differential equation, singular point, summability of series.

1. Let \( G = \{g_1, \ldots, g_n, \ldots\} \ (n \in \mathbb{N}) \) be the sequence of the strictly positive functions, defined and of the class \( C^\infty \) in \([r_0, 1]\), \( 0 \leq r_0 < 1 \). By \( \alpha \) we denote a positive constant.

Let us consider the differential operator \( D^\alpha_r(G) \) defined for the functions of the class \( C^\infty \) in \([r_0, 1]\) by the formulas

\[
\begin{align*}
D^0_r(G)f(r) &= f(r), \\
D^i_r(G)f(r) &= D^{i-1}_r(G)f(r) + (1-r)^\alpha g_i(r) \frac{d^i}{dr^i} D^{i-1}_r(G)f(r) \\
(\text{for } i = 1, 2, \ldots),
\end{align*}
\]

where \( r \in [r_0, 1] \).

In paper [7], applying the operator \( D_\alpha \), the following method of summability of the series introduced:

**Definition.** Let us consider the sequence \( F = \{f_1, \ldots, f_n, \ldots\} \) of the functions \( f_k \ (k \in \mathbb{N}) \) of the class \( C^\infty \) in \([r_0, 1]\). The numerical series \( \sum_{k=1}^{\infty} a_k \), where \( a_k \) are the real numbers, is summable to the sum \( s \) by the \( A_\alpha(n, G, F) \) method if the series \( \sum_{k=0}^{\infty} f^{(p)}(r)a_k \), \( (p = 0, 1, \ldots, n) \) are uniformly convergent in every interval \([r_0, \xi]\), \( r_0 < \xi < 1 \) and if

\[
\lim_{r \to 1^-} H_\alpha(n, G, F) = s,
\]

where

\[
H_\alpha(n, G, F, r) = \sum_{k=0}^{\infty} D^k_\alpha(G)f_k(r)a_k.
\]

It follows, from the foregoing definition and fromula (1), that
(3) \( H_\alpha(n,G,F,r) = H_\alpha(n-1,G,F,r) + (1-r)^\alpha g_\alpha(r) \frac{d}{dr} H_\alpha(n-1,G,F,r) \)
for \( r \in [r_0, 1) \).

The method \( A_\alpha(n,G,F) \) generalized some results of papers [2], [4], [5], [6].

In paper [7], the following theorem was shown:

**Theorem 1.** If a series \( \sum a_n \) is summable to sum \( s \) \((s \in \mathbb{R})\) by the method \( A_\alpha(n,G,F) \) then it is summable to sum \( s \) by the method \( A_\alpha(n-1,G,F) \).

In the present paper we will consider the case with \( s = +\infty \).

2. Let \( a, b \in \mathbb{R}, a < b \). We consider the differential equation

\[
(b-t)^\alpha u'(t) + [A + L(t)] u(t) = f(t), \quad t \in [a, b),
\]

where \( A \in \mathbb{R}, L, f \) are real functions defined in \([a, b)\).

We will prove the following:

**Theorem 2.** Assume that:

(i) \( A > 0 \),

(ii) \( \alpha \geq 1 \),

(iii) \( L \) is a continuous function defined in \([a, b]\) and \( L(b) = 0 \),

(iv) \( f \) is a continuous function defined in \([a, b]\) and

\[
\lim_{t \to b^-} f(t) = +\infty.
\]

Then for any solution \( u \) of equation (4), defined in \([t_0, b)\), the condition

\[
\lim_{t \to b^-} u(t) = +\infty,
\]

is satisfied.

**Proof.** Let \( u \) be a solution of equation (4), and \([t_0, b)\) be the domain of \( u \), where \( t_0 \in [a, b) \), and \( u(t_0) = u_0 \).

Then function \( u \) satisfies the equation

\[
u(t) = R(t, t_0) u_0 + \int_{t_0}^{t} R(t, s)(b-s)^{-\alpha} f(s) ds, \quad t \in [t_0, b),
\]

where
for \(a \leq s \leq t < b\).

Observe that for \(\alpha \geq 1\), \(\lim_{t \to b^-} R(t, t_0) u_0 = 0\).

Let \(M\) be any arbitrary positive number. From assumption (iv), if follows that there exists a real number \(t_1 \in (t_0, b)\) such that

(5) \[ f(t) \geq M, \quad t \in (t_1, b). \]

We have

\[
\int_{t_0}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds = \int_{t_0}^{t_1} R(t, s)(b - s)^{-\alpha} f(s)\,ds + \int_{t_1}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds = J_1(t) + J_2(t).
\]

Observe that

\[ J_1 = R(t, t_0) \int_{t_0}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds. \]

Since \(\lim_{t \to b^-} R(t, t_0) = 0\) then \(\lim_{t \to b^-} J_1(t) = 0\).

From (5), we get

\[ J_2 = R(t, t_0) \int_{t_1}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds \geq MR(t, t_0) \int_{t_1}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds. \]

By l'Hôpital's rule and assumption (iii), we obtain

\[
\lim_{t \to b^-} R(t, t_0) \int_{t_1}^{t} R(t, s)(b - s)^{-\alpha} f(s)\,ds = \lim_{t \to b^-} \frac{1}{A + L(t)} = \frac{1}{A},
\]

Hence

\[ \lim_{t \to b^-} J_2(t) \geq \frac{M}{A}. \]

By the fact that \(M\) is an arbitrary positive number, we have

\[ \lim_{t \to b^-} J_2(t) = +\infty. \]
Consequently
\[ \lim_{t \to b^-} u(t) = +\infty. \]

**Remark 1.** If assumption (iv) of Theorem 2 will be replace by the assumption
\[ \lim_{t \to b^-} f(t) = -\infty \]
then
\[ \lim_{t \to b^-} u(t) = -\infty. \]

**Remark 2.** The condition \( \alpha \geq 1 \) in Theorem 2 is essential. For an example, let \( 0 < \alpha, \beta < 1 \) be such that \( \alpha + \beta < 1 \).

Considering the equation
\[(1 - t)^{-\alpha} u'(t) + (1 - \alpha) u(t) = (1 - t)^{-\beta},\]
we get the solution of this equation in the from
\[ u(t) = C \exp((1 - t)^{-\alpha}) + \int_{t_0}^{t} (1 - s)^{-\alpha + \beta} \exp((1 - t)^{-\alpha} - (1 - s)^{-\alpha}) ds, \]
where \( C = \exp(-(1 - t_0)^{-\alpha}) \).

Consequently,
\[ \lim_{t \to 1^-} u(t) = C + \int_{t_0}^{t} (1 - s)^{-\alpha + \beta} \exp(-(1 - s)^{-\alpha}) ds < +\infty. \]

However,
\[ \lim_{t \to b^-} f(t) = \lim_{t \to 1^-} (1 - t)^{-\beta} = +\infty. \]

3. We will study the problem of summability of the series, which we considered at the first part of this paper.

Assume that
\[ \lim_{r \to 1^-} H_a(n, G, F, r) = +\infty \]
and let
\[ p(r) := H_a(n, G, F, r). \]

Observe that \( p \) is the continuous function in \([r_0, 1]\). Further, the function \( u \), defined by the formula:
\[ u(r) := H_a(n - 1, G, F, r), \quad r \in [r_0, 1), \]
satisfies the equation
(6) \[(1 - r)^\alpha g_n(r)u'(r) + u(r) = p(r).\]

Define a function \(h\) by the formula
\[
h(r) := \frac{1}{g_n(r)} - \frac{1}{g_n(1)}.
\]

It is easy to see that the function \(h\) is continuous in \([r_0, 1]\) and \(h(1) = 0\).

The above formula implies that equation (6) is equivalent to the following
\[(7) \quad (1 - r)^\alpha u'(r) + [A + h(r)]u(r) = \frac{p(r)}{g_n(r)},\]

where \(A = \frac{1}{g_n(1)} > 0\).

Applying Theorem 2, we have
\[
\lim_{r \to 1^-} H_{\alpha}(n - 1, G, F, r) = \lim_{r \to 1^-} u(r) = \lim_{r \to 1^-} \frac{p(r)}{g_n(r)} = +\infty.
\]

From the foregoing formula and the results of paper [7], we get

**Theorem 3.** If a series \(\sum a_n\) is summable to sum \(s \in \mathbb{R}\) by the method \(A_{\alpha}(n, G, F)\) then it is summable to sum \(s\) by the method \(A_{\alpha}(n - 1, G, F)\).

**REFERENCES**


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Received on 11.06.1997 and, in revised form, on 12.02.1998.