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INTERPOLATION INEQUALITIES
IN ORLICZ-SOBOLEV SPACE

ABSTRACT: We consider the problem of determining upper bounds for norms of functions from Orlicz-Sobolev space $W^{1,M} (\Omega)$, $j = 0, 1, 2, ..., m$ in terms of norms of the space $W^{m,M} (\Omega)$ and Orlicz space $L^M (\Omega)$.

The interpolation inequalities of this type are well-known for classical Sobolev spaces $W^{m,p} (\Omega)$, $p \geq 1$, (see e.g. [6], [7] and also [1]).

KEY WORDS: Orlicz-Sobolev space, Orlicz space, modular, modular space.

Let $\mathcal{R}$ be a real vector space. A functional $\rho : \mathcal{R} \to [0, +\infty]$, where $\mathcal{R} = [0, +\infty]$, is called a convex pseudomodular in $\mathcal{R}$, if $\rho(0) = 0$, $\rho(-x) = \rho(x)$ and $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for $x, y \in \mathcal{R}$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. If, additionally, $\rho(x) = 0$ only for $x = 0$, then $\rho$ is called a convex modular on $\mathcal{R}$, (see [5]).

By a function $M : [0, +\infty) \to [0, +\infty)$ we mean a map which is convex, vanishing and continuous at zero and not vanishing everywhere. A function $M$ satisfies the condition $A_2$ if there exists a constant $K > 0$ such that $M(2u) \leq K M(u)$ for every $u > 0$ (for more details see e.g. [5]).

Let $\Omega$ be a nonempty open set in the $N$-dimensional real Euclidean space $\mathbb{R}^N$. By $X$ we shall denote the vector space of all locally integrable functions on $\Omega$ with equality almost everywhere on $\Omega$. Let $m$ be a fixed non-negative integer number. The convex modular $J$ on $X$ we define in the following manner

$$J(f) = \sum_{|\alpha| \leq m} \int_{\Omega} |\partial^\alpha f(x)| dx \quad \text{for} \quad f \in X,$$

where $D^\alpha f$ is the distributional derivative of $f$. By the Orlicz-Sobolev space $W^{m,M} (\Omega)$ we mean the set of all functions $f \in X$, possessing distributional derivatives $D^\alpha f$ up to order $m$, for which there exists a constant $a > 0$, depending of $f$, such that $J(\alpha f) < \infty$. The space $W^{m,M} (\Omega)$ equipped with the Luxemburg norm $\| \cdot \|_{W^{m,M}}$ generated by the modular $J$, is a Banach space, (see [2]). In the sequel we shall use $\| \cdot \|_{m}$ in place $\| \cdot \|_{W^{m,M}}$. Let $W^{m,M}_0 (\Omega)$ denote the closure of $C_0^\infty (\Omega)$ in the space $W^{m,M} (\Omega)$. 
In the Orlicz-Sobolev space $W^{m,M}(\Omega)$ we define the family functionals $(I_j)_{0 \leq j \leq m}$ as follows

$$I_j(f) = \sum_{|\alpha| = j} \int_{\Omega} M(|D^\alpha f(x)|)dx \quad \text{for} \quad f \in X.$$ 

For $j = 0$ the above functional is a convex modular and for $1 \leq j \leq m$ the functionals $I_j$ are convex pseudomodulars. Moreover we define the second family of functionals $(J_j)_{0 \leq j \leq m}$ by

$$J_j(f) = \sum_{|\alpha| = j} \int_{\Omega} M(|D^\alpha f(x)|)dx \quad \text{for} \quad f \in X.$$ 

The functionals $J_j$ are convex modulars. In particular, for $j = m$, we obtain the modular $J$ generating $W^{m,M}(\Omega)$.

The following theorem is proved in [4]:

**Theorem 1.** Let $\rho$ be a convex pseudomodular in a real vector space $\mathfrak{X}$ and let $\rho(cu) < \infty$ for some $c > 0$. If $\rho(a(u_n - u)) \to 0$ as $n \to \infty$ for every $a > 0$, then there exists $b > 0$ such that $\rho(bu_n) \to \rho(bu)$ as $n \to \infty$.

Let the mapping $f \to f^*$, for $f \in X$, denote zero extension of $f$ outside a set $\Omega$:

$$f^*(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^N - \Omega. \end{cases}$$

(1)

**Lemma 1.** Let $f \in W_0^{m,M}(\Omega)$. The mapping $f \to f^*$ specified by (1) is an isometric isomorphism of $W_0^{m,M}(\Omega)$ into $W^{m,M}(\mathbb{R}^N)$.

**Proof.** Let $f \in W_0^{m,M}(\Omega)$ and let $(\phi_n)_{n=1}^\infty \subset C_0^\infty(\Omega)$ be a sequence converging to $f$ in the space $W_0^{m,M}(\Omega)$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have for $|\alpha| \leq m$

$$\left| \int_{\Omega} f(x)D^\alpha \varphi(x)dx - \int_{\Omega} \phi_n f(x)D^\alpha \varphi(x)dx \right| \leq c \| f - \phi_n \|_{W^{m,M}(\Omega)}.$$

Since

$$\int_{\mathbb{R}^N} f^*(x)D^\alpha \varphi(x)dx = \lim_{n \to \infty} \int_{\Omega} \phi_n(x)D^\alpha \varphi(x)dx = \int_{\Omega} f(x)D^\alpha \varphi(x)dx,$$
\[= (-1)^{|\alpha|} \lim_{n \to \infty} \int_{\Omega} D^\alpha \phi_n(x) \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega} D^\alpha f(x) \varphi(x) dx =
\]
\[= (-1)^{|\alpha|} \int_{R^N} (D^\alpha f)^*(x) \varphi(x) dx,
\]
so \( D^\alpha f^* = (D^\alpha f)^* \) in the distributional sense on \( R^N \). Then we have
\[\int_{\Omega} M(|D^\alpha f(x)|) dx = \int_{R^N} M(|(D^\alpha f)^*(x)|) dx = \int_{R^N} M(|D^\alpha f^*(x)|) dx
\]
for \(|\alpha| \leq m\). Hence \( \|f\|_{W^{m,M}(\Omega)} = \|f^*\|_{W^{m,M}(R^N)} \).

This lemma, for the case \( M(u) = u^p, \ 1 \leq p < \infty \) can be found in [1].

Lemma 2. Let \( M \) satisfy the condition \( \Delta_2 \). Let \( -\infty \leq a < b \leq \infty \), and let \( 0 < \varepsilon_0 < \infty \). There exists a constant \( C = C(\varepsilon_0, M, b-a) \) for \( 0 < b-a < \infty \), such that for every function \( f \in C^2(a,b) \) and for every \( 0 < \varepsilon \leq \varepsilon_0 \)
\[\int_a^b M(|f'(t)|) dt \leq C \int_a^b M(\varepsilon(f''(t))) dt + C \int_a^b M(\varepsilon^{-1}|f(t)|) dt.
\]
If \( b-a = \infty \), then (2) holds with a constant \( C = C(M) \) for every \( \varepsilon > 0 \).

Proof. We assume, that \( \varepsilon_0 = 1 \) and \( 0 < b-a < \infty \). If \( \xi \in (a, a + (1/3)(b-a)) \) and \( \eta \in (a + (2/3)(b-a), b) \), then there exists \( \lambda \in (\xi, \eta) \) such that
\[|f'(\lambda)| = \left| \frac{f(\eta) - f(\xi)}{\eta - \xi} \right| \leq \frac{3}{b-a} (|f(\eta)| + |f(\xi)|).
\]
It follows, by convexity of \( M \), that for any \( x \in (a, b) \)
\[M(|f'(x)|) \leq \frac{1}{3} M\left( \frac{9}{b-a} |f(\xi)| \right) + \frac{1}{3} M\left( \frac{9}{b-a} |f(\xi)| \right) + \frac{1}{3} M\left( 3 \int_a^b |f''(t)| dt \right).
\]
Applying Jensen’s inequality and integrating the above inequality with respect to \( \xi \) over \( \xi \in (a, a + (1/3)(b-a)) \) and with respect to \( \eta \) over \( \eta \in (a + (2/3)(b-a), b) \), we obtain
\[M(|f'(x)|) \leq \frac{1}{b-a} \int_a^b M\left( \frac{9}{b-a} |f(t)| \right) + \frac{1}{b-a} \int_a^b M(3(b-a) |f''(t)|) dt.
\]
Integrating with respect to \( x \) over \( (a,b) \), we are led to
(3) \[ \int_a^b M( |f''(t)|) dt \leq \int_a^b M\left( \frac{9}{b-a} |f(t)| \right) dt + \int_a^b M(3(b-a)|f''(t)|) dt. \]

Since \( 0 < \varepsilon \leq 1 \) there exists a positive integer \( n \) such that \((1/2)\varepsilon \leq (1/n) \leq \varepsilon \). Setting \( a_j = a + (b-a)(j/n) \) for \( j = 0, 1, ..., n \), we obtain from (3), noting that \( a_{j+1} - a_j = (b-a)/n \),

\[ \int_a^b M( |f''(t)|) dt \leq \sum_{j=0}^{n-1} \left\{ \int_{a_j}^{a_{j+1}} M\left( \frac{18}{b-a} \varepsilon^{-1} |f(t)| \right) dt + \int_{a_j}^{a_{j+1}} M(3(b-a)\varepsilon |f''(t)|) dt \right\}. \]

Let \( p \) be a positive integer number such that

\[ \max\left\{ \frac{18}{b-a}, 3(b-a) \right\} \leq 2^p. \]

Then, applying the condition \( \Delta_2 \), we have

(4) \[ \int_a^b M( |f''(t)|) dt \leq K^p \int_a^b M(\varepsilon^{-1} |f(t)|) dt + K^p \int_a^b M(\varepsilon |f''(t)|) dt. \]

Let now \( \varepsilon_0 \) be an arbitrary positive number. For \( 0 < \varepsilon \leq \varepsilon_0 \) we have \( 0 < (\varepsilon/\varepsilon_0) \leq 1 \). Thus, from (4), we obtain

\[ \int_a^b M( |f''(t)|) dt \leq C \int_a^b M\left( \frac{\varepsilon_0}{\varepsilon} |f(t)| \right) dt + C \int_a^b M\left( \frac{\varepsilon}{\varepsilon_0} |f''(t)| \right) dt. \]

Suppose, that \( b-a = \infty \). To be specific we assume \( a < \infty \) and \( b = \infty \). For given \( \varepsilon > 0 \) let \( a_j = a + j\varepsilon \), \( j = 0, 1, 2, ... \). Using (3) we have

\[ \int_a^\infty M( |f''(t)|) dt = \sum_{j=0}^{\infty} \int_{a_j}^{a_{j+1}} M( |f''(t)|) dt \leq \int_a^\infty M(9\varepsilon^{-1} |f(t)|) dt + \int_a^\infty M(3\varepsilon |f''(t)|) dt. \]

By the condition \( \Delta_2 \)

\[ \int_a^\infty M( |f''(t)|) dt \leq C \int_a^\infty M(\varepsilon^{-1} |f(t)|) dt + C \int_a^\infty M(\varepsilon |f''(t)|) dt, \]

which is the desired inequality, where the constant \( C \) depends only on \( M \).
The other possibilities are similar.

**Lemma 3.** Let $M$ satisfy the condition $\Delta_2$. Let $0 < \delta_0 < \infty$, let $m \geq 2$ and let $\varepsilon_0 = \min\{\delta_0, \delta_0^2, ..., \delta_0^{m-1}\}$. Suppose that there exists a constant $K = K(\delta_0, M, \Omega)$ such that for every $\delta$, $0 < \delta \leq \delta_0$ and for every $u \in W^{2,M}(\Omega)$ we have

$$I_1(u) \leq K I_2(\delta u) + K I_0(\delta^{-1} u).$$

Then there exists a constant $C = C(\varepsilon_0, m, M, \Omega)$ such that for every $0 < \varepsilon \leq \varepsilon_0$, every integer $j$, $0 \leq j \leq m-1$, and every $u \in W^{m,M}(\Omega)$, we have

$$I_j(u) \leq C I_m(\varepsilon u) + K I_0\left(\varepsilon^{m-j} u\right).$$

**Proof.** For $j = 0$ the inequality (6) is obvious. We consider only the case $1 \leq j \leq m-1$. We first prove (6) for $j = m-1$ by induction on $m$. Then for $m = 2$ the inequality (6) is exactly (5). Assume, that (6) holds for some $k$, $2 \leq k \leq m-1$,

$$I_{k-1}(u) \leq K_1 I_k(\delta u) + K_1 I_0(\delta^{-(k-1)} u)$$

for every $0 < \delta \leq \delta_0$ and $u \in W^{k,M}(\Omega)$. Let $u \in W^{k+1,M}(\Omega)$ and let $|\alpha| = k-1$. Then for $u \in W^{k+1,M}(\Omega)$ we have $D^\alpha u \in W^{2,M}(\Omega)$. Thus, from (5) we obtain

$$I_1(D^\alpha u) \leq K_2 I_2(\delta D^\alpha u) + K_2 I_0(\delta^{-1} D^\alpha u).$$

Then, by (7), we have for $0 < \eta \leq \delta_0$

$$I_k(u) \leq \sum_{|\alpha|=k-1} I_1(D^\alpha u) \leq K_3 I_{k+1}(\delta u) + K_3 I_{k-1}(\delta^{-1} u) \leq K_3 I_{k+1}(\delta u) + K_1 K_3 I_k(\delta^{-1} \eta u) + K_1 K_3 I_0(\delta^{-1} \eta^{1-k} u).$$

We may assume without loss of generality, that $2 K_1 K_3 \geq 1$. Taking $\eta = \delta/2 K_1 K_3$ and applying $\Delta_2$, we obtain

$$I_k(u) \leq K_4 I_{k+1}(\delta u) + K_4 I_0(\delta^{-k} u).$$

This completes the induction establishing (6) for $j = m-1$ with $\varepsilon = \delta$. 
By induction on \( j \) we prove

\[
I_j(u) \leq K_3 I_m(\delta^{m-j} u) + K_3 I_0(\delta^{-j} u)
\]

for \( 1 \leq j \leq m-1 \) and \( 0 < \delta \leq \delta_0 \). Setting \( k = m \) in (7) we obtain (9) in the special case \( j = m-1 \). Thus for \( j = m-1 \) (9) holds. Assume, therefore, that (9) holds for some \( j \), \( 2 \leq j \leq m-1 \). We prove that it also holds for \( j-1 \). From (7) and (8) we obtain

\[
I_{j-1}(u) \leq K_6 I_m(\delta^{m-(j-1)} u) + K_6 I_0(\delta^{1-j} u).
\]

Thus (9) holds. Let \( 0 < \varepsilon \leq \min\{\delta_0, \delta_0^2, \ldots, \delta_0^{m-1}\} \) be arbitrary. Then \( \varepsilon \leq \delta_0^{m-j} \) for every \( j = 1, 2, \ldots, m-1 \). Hence \( \varepsilon^{\frac{1}{(m-j)}} \leq \delta_0 \). Now (6) follows by setting \( \delta = \varepsilon^{\frac{1}{(m-j)}} \) in (9).

**Theorem 2.** Let \( M \) satisfy the condition \( \Delta_2 \). There exists a constant \( K = K(m, M, N) \) such that for any \( \varepsilon > 0 \), any integer \( j \), \( 0 \leq j \leq m-1 \), and any \( u \in W_0^{m, M}(\Omega) \)

\[
I_j(u) \leq K I_m(\varepsilon u) + K I_0\left(\frac{\varepsilon}{\varepsilon - j} u\right).
\]

**Proof.** The operator \( Tu = u^* \), \( u \in W_0^{m, M}(\Omega) \), specified by (1) is, by Lemma 1, an isometric isomorphism of \( W_0^{m, M}(\Omega) \) into \( W_0^{m, M}(R^N) \). Thus it is sufficient to prove the theorem for \( \Omega = R^N \). By Lemma 3 we need consider only the case \( j = 1, m = 2 \). For \( j = 0, m = 1 \) the desired thesis is obvious.

Let \( \varepsilon > 0 \) be arbitrary and let \( \phi \in C_0^\infty(R^N) \). By Lemma 2 we have

\[
\int_{R^N} M(||D_j \phi(x)||) dx_j \leq K \int_{R^N} M(\varepsilon ||D_j^2 \phi(x)||) dx_j + K \int_{R^N} M(\varepsilon^{-1} ||\phi(x)||) dx_j.
\]

Integrating the above inequality with respect to the remaining components of \( x \), we obtain

\[
\int_{R^N} M(||D_j \phi(x)||) dx \leq K \int_{R^N} M(\varepsilon ||D_j^2 \phi(x)||) dx + K \int_{R^N} M(\varepsilon^{-1} ||\phi(x)||) dx.
\]

Hence

\[
I_1(\phi) = \sum_{j=1}^N \int_{R^N} M(||D_j \phi(x)||) dx \leq K \sum_{|\alpha| = 2} \int_{R^N} M(\varepsilon ||D^\alpha \phi(x)||) dx +
\]
\[ + KN \int_{\mathbb{R}^N} M(\varepsilon^{-1}|\phi(x)|) dx \leq K_1 I_1(\varepsilon\phi) + K_1 I_0(\varepsilon^{-1}\phi). \]

Let \( u \in W_{0}^{m,M}(\Omega) \). Since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( W^{m,M}(\mathbb{R}^N) \), (see [3]), it follows that there exists a sequence \((\phi_n) \in C_0^\infty(\mathbb{R}^N)\) such that \( J(a(u - \phi_n)) \to 0 \) as \( n \to \infty \) for every \( a > 0 \). Hence \( I_i(a(u - \phi_n)) \to 0 \) as \( n \to \infty \) for every \( a > 0 \) and \( i = 0, 1, 2 \). By Theorem 1 and the condition \( \Delta_2 \) we have

\[ I_0(\varepsilon^{-1}\phi_n) \leq I_0(\varepsilon^{-1}u), \quad I_1(\phi_n) \to I_1(u) \text{ and } I_2(\varepsilon\phi_n) \leq I_2(\varepsilon u) \text{ as } n \to \infty. \]

Using (10) we have

(12) \[ I_1(\phi_n) \leq K_1 I_2(\varepsilon\phi_n) + K_1 I_0(\varepsilon^{-1}\phi_n) \quad \text{for} \quad n = 1, 2, \ldots. \]

Let \( n \to \infty \) in (12). Then, we obtain

\[ I_1(u) \leq K_1 I_2(\varepsilon u) + K_1 I_0(\varepsilon^{-1}u). \]

This completes the proof.

**Theorem 3** Let \( \Omega \) be arbitrary open set in \( \mathbb{R}^N \) and let \( M \) satisfy the condition \( \Delta_2 \). Then there exists a constant \( K = K(m, M, N) \) such that for \( 0 \leq j \leq m \) and any \( u \in W_{0}^{m,M}(\Omega) \)

\[ \|u\|_j \leq K \|u\|_{m}^{j/m} \|u\|_0^{(m-j)/m}. \]

**Proof.** For \( j = 0 \) and \( j = m \) the desired inequality is obvious. Let \( 0 < j < m \). By means of Theorem 2 for \( 0 < \varepsilon \leq 1 \) we have

(13) \[ J_j(u) = \sum_{i=0}^{j} I_i(u) \leq \sum_{i=0}^{j} \left\{ K I_m(\varepsilon u) + K I_0\left(\varepsilon^{-\frac{i}{m-i}}u\right) \right\} \leq K_1 J_m(\varepsilon u) + K_1 J_0\left(\varepsilon^{-\frac{i}{m-i}}u\right) \]

for any \( u \in W_{0}^{m,M}(\Omega) \). It follows by continuity of \( M \) that

\[ J_m\left(\frac{u}{\|u\|_{m}}\right) \leq 1 \quad \text{and} \quad J_0\left(\frac{u}{\|u\|_0}\right) \leq 1 \]
for any \( u \in W_0^{m,M}(\Omega), u \neq 0 \). Now we set \( \varepsilon = (\|u\|_0/\|u\|_m)^{(m-j)/m} \) and denote \( B = \|u\|_m^{j/m} \|u\|_0^{(m-j)/m} \). Then, by (13), we obtain for fixed \( j \), \( J_j(u/B) \leq 2K_1 \). We may assume that \( 2K_1 \geq 1 \). Then, by convexity of \( J_j \) we have

\[
\|u\|_j \leq 2K_1 \|u\|_m^{j/m} \|u\|_0^{(m-j)/m}.
\]

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