ON THE HOMOGENEOUS QUASI POLYHARMONIC POLYNOMIALS

ABSTRACT: In this paper the effective construction of maximal system linearly independent homogeneous quasi polyharmonic polynomials of degree $n$ of $q$ independent variables is given.

KEY WORDS: polyharmonic equations, quasi polyharmonic polynomials, linear independence.

1. In the present paper we shall give the construction of the system of homogeneous quasi polyharmonic polynomials of degree $n$ and depending on $q$ independent variables of the form

$$P_n^q(x_1, x_2, ..., x_q) = \sum_{p_1+p_2+...+p_q=n} a_{p_1p_2...p_q} x_1^{p_1} x_2^{p_2} ... x_q^{p_q},$$

where $p_i, i=1,2,...,q$, are nonnegative integers. The polynomials (1) satisfy, the quasi $p$-harmonic equation

$$L^p u = 0, \quad p \in N,$$

where

$$L u = \sum_{i,j=1}^q a_{ij} D_{x_i x_j}^2 u, \quad L^p = L(L^{p-1}),$$

and

$$A = [a_{ij}], \quad a_{ij} \in R, \quad i, j = 1,2,...,q$$

is the positive definite matrix.

In the monograph [1] the similar problem for $p=1, q=3$ and for the diagonal matrix $A$ is solved. The problem is considered by the complicated method by curvilinear coordinates defined by the family of convenient quadrics.

In the paper [4] the similar problem for the symmetric matrix $A$ by another method is solved.

2. Let us consider the case in which $p=1, q \in N, n \in N$, and the equation

$$Lu = 0.$$
Let us consider the positive definite quadratic from

\[ F(x) = \sum_{i,j=1}^{q} a_{ij} x_i x_j. \]  

(5)

Let

\[ C = [C_{ij}], \quad i, j = 1, 2, \ldots, q, \]

denote a nonsingular matrix such that the transformation

\[ x_i = \sum_{j=1}^{q} C_{ij} t_j, \quad i = 1, \ldots, q \]

(6)

transforms the quadratic from (5) to the canonical from

\[ \sum_{i=1}^{q} t_i^2. \]

(7)

Let

\[ t_i = \sum_{j=1}^{q} k_{ij} x_j, \quad i = 1, \ldots, q \]

(T−1)

denote the inverse transformation to the transformation (T). It is known [2], that there exist exactly

\[ h(n, q, 1) = \binom{n + q - 1}{q - 1} - \binom{n + q - 3}{q - 1} \]

linearly independent harmonic homogeneous polynomials of degree \( n \) of \( q \) independent variables \( t_1, \ldots, t_q \) of the form

\[ W_{i}^{n,1}(t) = \sum_{p_1 + p_2 + \ldots + p_q = n} b^{p_1 p_2 \ldots p_q} t_1^{p_1} t_2^{p_2} \ldots t_q^{p_q}, \quad i = 1, \ldots, h(n, q, 1), \]

(8)

where \( p_i, \quad i = 1, \ldots, q, \) are nonegative integers and \( t = (t_1, \ldots, t_q) \).

Let \( x = (x_1, \ldots, x_q) \) and let

\[ r^2(x) = \sum_{i=1}^{q} \left( \sum_{j=1}^{q} k_{ij} x_j \right)^2, \quad R^2(t) = \sum_{i=1}^{q} t_i^2 \]

3. Now we shall prove the following

**Lemma 1.** There exist exactly \( h(n, q, 1) \) linearly independent quasi harmonic homogeneous polynomials \( Q_{i}^{n,1}(x), \quad i = 1, \ldots, h(n, q, 1) \) satisfying equation (3a).
PROOF. Let $M$ denotes the number of the polynomials $Q^n_{i,1}(x)$ linearly independent. Let us assume that $M < h(n,q,1)$. It is known [3] that the polynomials $W^n_{j,1}(t), \ j=1,2,...,M$ satisfy the equation (3a) with respect to variable $t$, and it is known that the condition

\begin{equation}
\sum_{i=1}^{h(n,q,1)} C_i W^n_{i,1}(t) = 0
\end{equation}

implies the conditions $C_i = 0, \ i=1,...,h(n,q,1)$. Let us suppose, that by identity

\begin{equation}
\sum_{j=1}^{M} m_j Q^n_{j,1}(t) = 0
\end{equation}

imply that $m_j = 0, \ j=1,...,M$.

By the transformation $(T^{-1})$ we get

\begin{equation}
\sum_{j=1}^{M} m_j W^n_{j,1}(t) = 0
\end{equation}

and $m_j = 0, \ j=1,2,...,M$ what is contradics with the conditions (9). Similarly for $M > h(n,q,1)$ we obtain the contradiction. Therefore $M = h(n,q,1)$.

4. Let us consider the system of the polynomials

\begin{equation}
W^n_{i,1}(t) = H^n_i(t), \ i=1,...,h(n,q,1),
\end{equation}

where $H^n_i(t), \ i=1,2,...,h(n,q,1)$, are harmonic linearly independent homogeneous polynomials of the degree $n$ of $q$ variables, $t_i, \ i=1,2,...,q$.

\begin{equation}
W^n_{i,2}(t) = R^2(t) H^n_{i,2}(t), \ i=1,2,...,h(n-2,q,1),
\end{equation}

where $H^n_{i,2}(t), \ i=1,2,...,h(n-2,q,1)$, are the harmonic homogeneous polynomials of the degree $n-2$ linearly independent of $q$ variables $t_i, \ i=1,2,...,q$, and so one

\begin{equation}
W^n_{i,p}(t) = R^{2p-2}(t) H^n_{i,2p+2}(t), \ i=1,2,...,h(n-2p+2,q,1),
\end{equation}

where $H^n_{i,2p+2}(t), \ i=1,2,...,h(n-2p+2,q,1)$, are the harmonic homogeneous polynomials of the degree $n-2p+2$ linearly independent, of the $q$ variables $t_i, \ i=1,...,q$. 


The number of all the polynomials of the system (10), \( i = 1, \ldots, p \) is equal 
\[
h(n, q, 1) + h(n - 2, q, 1) + \ldots + h(n - 2p + 2, q, 1) = h(n, q, p).
\]

By the paper [2] we shall give the following.

**Lemma 2.** The system of the polynomials \( W_i^{n,p}(t) \), \( i = 1, \ldots, p \), is the system of the linearly independent polynomials.

**Proof.** Let

\[
W(t) = \sum_{i=1}^{p} W_i(t)
\]

where

\[
W_1(t) = \sum_{i=1}^{h(n, q, 1)} C_i^{(1)} W_i^{n,1}(t),
\]

\[
W_2(t) = \sum_{i=1}^{h(n, q, 1)} C_i^{(2)} W_i^{n,2}(t),
\]

and so one

\[
W_p(t) = \sum_{i=1}^{h(n-2p+2, q, 1)} C_i^{(p)} W_i^{n,p}(t),
\]

where \( C_i^{(j)} \) are the constants.

By the [3], p. 196 and by (10), \( j = 1, 2, \ldots, p \), we obtain

\[
\Delta^p W(t) = 0
\]

and

\[
\Delta^{p-1} W(t) = \Delta^{p-1} W_p(t) = A(n, q, p) \sum_{i=1}^{h(n-2p+2, q, 1)} C_i^{(p)} H_i^{n-2p+2}(t) = 0,
\]

where \( A(n, q, p) \) is a constant. By the linearly independent of the polynomials \( H_i^{n-2p+2} \) we obtain the following conditions

\[
C_i^{(p)} = 0, \quad i = 1, \ldots, h(n - 2p + 2, q, 1).
\]

Similarly we obtain

\[
\Delta^{p-2} W(t) = \Delta^{p-2} W_{p-1}(t) = A_i(n, q, p) \sum_{i=1}^{h(n-2p+4, q, 1)} C_i^{(p-1)} H_i^{n-2p+4}(t) = 0.
\]
From above conditions we have

$$C_i^{(n-1)} = 0, \quad i = 1, 2, \ldots, h(n - 2p + 4, q, l).$$

In and we obtain similarly the equations

$$C_i^{(1)} = 0, \quad i = 1, 2, \ldots, h(n, q, l).$$

which completes the proof of Lemma 2.

5. Now we shall give the system $h(n, q, p)$ of the quasi $p$-harmonic homogeneous polynomials $Q_i^{n,p}(x)$ of degree $n$. Let us consider the system of the quasi $p$-harmonic polynomials

$$Q_i^{n,1}(x), \quad i = 1, \ldots, h(n, q, l),$$

$$Q_i^{n,2}(x) = r^2(x)Q_i^{n-2,1}(x), \quad i = 1, \ldots, h(n - 2, q, l),$$

$$Q_i^{n,p}(x) = r^{2p-2}(x)Q_i^{n-2p+2,1}(x), \quad i = 1, \ldots, h(n - 2p + 2, q, l),$$

By the lemmas 1, 2, we obtain following

**Theorem.** The system of the polynomials (15) which number is equal $h(n, q, p)$ is of maximal system linearly independent homogeneous quasi $p$-harmonic polynomials of the degree $n$, of the $q$ variables $x_1, \ldots, x_q$.

**References**


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