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PROXIMATE TYPE IN REFERENCE TO GENERALIZED BIAxisymmetric POTENTIALS

ABSTRACT: Let \( F^{(\alpha, \beta)}(x, y) \) be a real valued regular solution to the generalized biaxially symmetric potential equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + \frac{2\beta + 1}{y} \frac{\partial}{\partial y} \right) F^{(\alpha, \beta)} = 0, \quad \alpha > \beta > -\frac{1}{2}.
\]

To obtain a more refined measure of growth then is given by [1] an approximation theorem for arbitrary proximate types and some more asymptotic properties have been proved. The proximate type is constructed for a class of Generalized Biaxially Symmetric Potential (GBASP). Lastly, we obtain lower and upper bounds for proximate type in reference to growth parameters of GBASP.

KEY WORDS: generalized biaxially symmetric potentials, proximate type, Cauchy data, regular growth.

1. INTRODUCTION

Let \( F^{(\alpha, \beta)} = F^{(\alpha, \beta)}(x, y) \) be a real valued regular solution to the generalized biaxially symmetric potential equation

\[
(1.1) \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + \frac{2\beta + 1}{y} \frac{\partial}{\partial y} \right) F^{(\alpha, \beta)} = 0, \quad \alpha > \beta > -\frac{1}{2}.
\]

subject to the Cauchy data

\[
F^{(\alpha, \beta)}(0, y) = F^{(\alpha, \beta)}(x, 0) = 0
\]

along the singular lines in open hypersphere \( \sum_{\beta}^{(\alpha, \beta)} : x^2 + y^2 < R^2 \). These solutions, called the generalized biaxially symmetric potentials (GBASP) can be expanded in \( \sum_{\alpha}^{(\alpha, \beta)} \) uniquely as

\[
(1.2) \quad F^{(\alpha, \beta)}(x, y) = \sum_{n=0}^{\infty} a_n R^{(\alpha, \beta)}_n(x, y), \quad x, y \in R \quad \text{(set of real numbers)}
\]

in terms of the complete set

\[
(1.3) \quad R^{(\alpha, \beta)}_n(x, y) = P^{(\alpha, \beta)}_n [(x^2 - y^2)/(x^2 + y^2)]/P^{(\alpha, \beta)}_n(1),
\]

of biaxisymmetric harmonic potentials, where \( P^{(\alpha, \beta)}_n \) are Jacobi polynomials.
Now we consider a positive function \( T(r) \) in \( 0 < r < R, \ 0 < R < \infty \), having the properties:

(i) \( T(r) \rightarrow T \) as \( r \rightarrow R, \ 0 \leq T \leq \infty; \)

(ii) \( \frac{(R-r)T''(r)}{RT(r)} \rightarrow 0 \) as \( r \rightarrow R \)

where \( T''(r) \) denotes the derivatives of \( T(r) \). Such a function \( T(r) \) is called the proximate type.

2. ASYMPTOTIC PROPERTIES

**Theorem 1.** For every continuously differentiable proximate type \( T(r) \), there exists a twice continuously differentiable proximate type \( S(r) \) such that

\[
\lim_{r \to R} \frac{(R-r)^2 \log((R-r)/R)S''(r)}{R^2 S(r)} = 0,
\]

and

\[
T(r) \equiv S(r) \quad \text{as} \quad r \to R.
\]

**Proof.** Let us assume that \( S(r) \) be a proximate type and coincide with \( T(r) \) on the sequence \( \{r_n\} \) in \([0, R)\) as

\[
T(r_n) = S(r_n), \quad r_n = R \left(1 - \frac{1}{4^n}\right), \quad n = 0, 1, 2, \ldots.
\]

In this case, for \( r \) lying in the intervals \([r_n, r_{n+1})\)

\[
\log \frac{T(r)}{S(r)} = \left| \int_{r_n}^{r} \left[ \frac{T'(x)}{T(x)} - \frac{S'(x)}{S(x)} \right] dx \right| = \left| \int_{r_n}^{r} o \left( \frac{R-x}{R-x} \right) dx \right| = o \left( \log \frac{R-r}{R-r_n} \right) = o(1) \quad \text{as} \quad r \to R.
\]

Hence \( T(r) = S(r) e^{o(1)} \), which implies (2.2).

Thus it is sufficient to construct a twice continuously differentiable proximate types \( S(r) \) satisfying the conditions (2.1) and (2.3). Define the functions on the interval \([0, 3/4] \):
\[ \phi = \begin{cases} t & 0 \leq t \leq 1/4, \\ -2t + 3/4 & 1/4 \leq t \leq 1/2, \\ t - 3/4 & 1/2 \leq t \leq 3/4, \end{cases} \]

and

\[ \xi(\sigma) = \int_0^\sigma \phi(t) \, dt. \]

Since \( \phi(t) \) is continuous on \([0,3/4]\), it follows that \( \xi(\sigma) \) is continuously differentiable on \([0,3/4]\). We also have

(a) \( 0 = \xi(0) = \xi(3/4) = \xi'(0) = \xi'(3/4) \),

(b) \( 0 \leq \xi(\sigma) \leq 3/16 \),

(c) \( |\xi'(\sigma)| \leq 1/4 \),

(d) \( \int_0^{3/4} \xi(x) \, dx = \delta > 0 \).

Define a sequence \( \{\varepsilon_n\} \) such that

\[ \varepsilon_n = \frac{\log \left( T(r_{n+1})/T(r_n) \right)}{\delta}. \]

Since

\[ \frac{T'(r)}{T(r)} = o \left( \frac{R}{R-r} \right), \]

it gives

\[ \int_{r_n}^{r_{n+1}} \frac{T'(r)}{T(r)} \, dr = \int_{r_n}^{r_{n+1}} o \left( \frac{R}{R-r} \right) \, dr, \]

or

\[ \log \frac{T(r_{n+1})}{T(r_n)} = o(\log 4). \]

Hence

\[ \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{(Taking} \ \delta = \log 4). \]

Finally, we define

\[ \log S(r) = \log T(r_n) + \frac{\varepsilon_n R}{R-r_n} \int_{r_n}^r \xi \left( \frac{t-r_n}{R-r_n} \right) \, dt \]

on the interval \([r_n, r_{n+1}]\).
The varification of properties (i) and (ii) and derivation of (2.1) for the positive function $S(r)$ in (2.4) can be easily obtain.

**Theorem 2.** Let $T(r)$ be a proximate type. Then, for $\rho$ $(0 < \rho < \infty)$ and $T$ $(0 \leq T < \infty)$,

(a) $\exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right)$ is monotonically increasing for $r > r_0$.

(b) $\exp \left( \left( \frac{R-r-\rho}{R-r} \right) T(r + \rho) + \rho T \right) \equiv \exp \left( \left( \frac{R-r}{R} \right) T(r) \right)$ as $r \to R$.

**Proof.** (a) We have

$$\frac{d}{dr} \left[ \exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right) \right] = \frac{\exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right)}{(R/R-r)^{\rho+1}} \left[ \rho T(r) + \left( \frac{R-r}{R} \right) T'(r) \right].$$

For $T > 0$, the condition (ii) may be replaced by $((R-r)/R)T'(r) \to 0$ as $r \to R$. Thus, we get asymptotically

$$\frac{d}{dr} \left[ \exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right) \right] \geq \frac{\exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right)}{(R/R-r)^{\rho+1}} (\rho T - \varepsilon), \quad (0 < \varepsilon < \rho T).$$

In case, $T = 0$, $(R/(R-r))^\rho T(r) \to \infty$ as $r \to R$ and

$$\frac{d}{dr} \left[ \exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right) \right] = \frac{T(r) \exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right)}{(R/R-r)^{\rho+1}} \left( \rho + \frac{(R-r)T'(r)}{RT(r)} \right) >$$

$$> \frac{T(r) \exp \left( \left( \frac{R}{R-r} \right)^\rho T(r) \right)}{(R/R-r)^{\rho+1}} (\rho - \varepsilon).$$

(b) For $T > 0$, considering the function

(2.5) $L(r) = \exp \{(R-r/R) (T(r) - T)\}$. 

Thus, for all values of $r \to R$,

$$\frac{L'(r)}{L(r)} = o(1)$$

or

(2.6) \[ \lim_{r \to R} \log \frac{L(r + \rho)}{L(r)} = 0. \]

In case, $T = 0$, (2.5) reduced to

$$\frac{L'(r)}{L(r)} = T(r) \left[ \frac{(R-r)T'(r)}{RT(r)} - 1 \right].$$

In view of properties (i) and (ii), again $L'(r)/L(r) \to 0$ as $r \to R$ and (2.6) is available which means

$$L(r + \rho) \equiv L(r) \quad \text{as} \quad r \to R.$$  

This immediately correspond to (b).

3. CONSTRUCTION AND BOUNDS

Let a real valued GBASP $F^{(a,\beta)}$ regular in $\sum^{(a,\beta)}$ having order $\rho$ ($0 < \rho < \infty$), type $T$ ($0 \leq T \leq \infty$) and satisfying in addition (i) and (ii). Then for a given $n$ ($0 < n < \infty$), $T(r)$ satisfies also:

(iii) $T(r)$ is continuous and piecewise differentiable for $r > r_0$;

and

(iv) $\limsup_{r \to R} \frac{M(r,F^{(a,\beta)})}{\exp\{(R/r-r)^{\rho} T(r)\}} = n, \quad M(r,F^{(a,\beta)}) = \max_{x^2+y^2=r^2} |F^{(a,\beta)}(x,y)|.$

Now $T'(r)$ in (ii) can be interpreted as $T'(r^+)$ or $T'(r^-)$ whenever these are unequal and the comparison function $T(r)$ is called the proximate type of the given real valued GBASP. The existence of such comparison function established in [1]. Obviously, proximate type of a real valued GBASP is not uniquely determined. For example, if we add $\gamma/(R/r-r)^{\rho}, \quad 0 < \gamma < \infty$ in the proximate type $T(r)$ we, again obtain a new proximate type for the same GBASP and the corresponding value of $n$ is divided by $e^\gamma$.

The GBASP are natural extensions of harmonic or analytic functions. Hence, we anticipate properties, similar to those of the harmonic functions found from associated analytic $f$, by taking Ref, the real part of $f$. 
By the Hadamard three circle theorem, we know, if \( f(z) \) is analytic in finite disc, \( \log M(r, f) \) in an increasing convex function of \( \log r \) in \( 0 < r < R \). Using above theorem for \( F^{(\alpha, \beta)}(x, y) \) we have if \( F^{(\alpha, \beta)}(x, y) \) is regular in open hypersphere \( \sum_{R}^{(\alpha, \beta)} \), \( \log M(r, F^{(\alpha, \beta)}) \) is an increasing convex function of \( \log r \) in \( 0 < r < R \).

Moreover, it has the representation

\[
(3.1) \quad \log M(r, F^{(\alpha, \beta)}) = \log M(r_0, F^{(\alpha, \beta)}) + \int_{r_0}^{r} \frac{w(x, F^{(\alpha, \beta)})}{x} dx, \quad 0 < r_0 < r < R,
\]

where \( w(x, F^{(\alpha, \beta)}) \) is a positive, continuous and piecewise differentiable function of \( x \).

The existence of (3.1) established by using ([2], Lemma 1, eqs. 2.1, 2.2). Now we prove:

**Lemma 1.** For a real valued GBASP \( F^{(\alpha, \beta)} \) regular in open hypersphere \( \sum_{R}^{(\alpha, \beta)} \) and having order \( \rho \) and lower order \( \lambda \), we have

\[
(3.2) \quad \liminf_{r \to R} \frac{(R-r)w(r, F^{(\alpha, \beta)})}{R \log M(r, F^{(\alpha, \beta)})} \leq \lambda < \rho \limsup_{r \to R} \frac{(R-r)w(r, F^{(\alpha, \beta)})}{R \log M(r, F^{(\alpha, \beta)})},
\]

**Proof.** Let \( R_+ \) be the set of extended positive real numbers. Then, for \( A \in R_+ \cup \{0\} \), we define

\[
(3.3) \quad \limsup_{r \to R} \frac{w(r, F^{(\alpha, \beta)})}{r(R/R-r)} = A.
\]

For \( A = 0 \), \( \rho = 0 \). On differentiation (3.1) gives

\[
(3.4) \quad \frac{M'(r, F^{(\alpha, \beta)})}{M(r, F^{(\alpha, \beta)})} = \frac{w(r, F^{(\alpha, \beta)})}{r}.
\]

The expression (3.3) together with (3.4) is rewritten as

\[
\limsup_{r \to R} \frac{(R-r)M'(r, F^{(\alpha, \beta)})}{RM(r, F^{(\alpha, \beta)}) \log M(r, F^{(\alpha, \beta)})} = A.
\]

For given \( \varepsilon > 0 \) and \( r > r_0(\varepsilon) \),

\[
\frac{M'(r, F^{(\alpha, \beta)})}{M(r, F^{(\alpha, \beta)}) \log M(r, F^{(\alpha, \beta)})} < \frac{A + \varepsilon}{(R-r)/R}.
\]
Integrating above inequality, we get
\[
\log \log M(r, F^{(\alpha, \beta)}) < o(1) + (A + \varepsilon) \log (R/R - r).
\]
Passing to limits, we get
\[
\lambda \leq \lim_{r \to R} \frac{(R - r) w(r, F^{(\alpha, \beta)})}{R r \log M(r, F^{(\alpha, \beta)})},
\]
which holds for \( A = \infty \). Likewise, for lower order \( \lambda \),
\[
\lambda \geq \liminf_{r \to R} \frac{(R - r) w(r, F^{(\alpha, \beta)})}{R r \log M(r, F^{(\alpha, \beta)})}.
\]
Combining above two inequalities (3.2) is immediate.

**Lemma 2.** If a real valued GBASP \( F^{(\alpha, \beta)} \) regular in open hypersphere \( \Sigma^{(\alpha, \beta)}_R \) having order \( \rho (0 < \rho < \infty) \), type \( T \) and lower type \( t \) then

\[
\text{(3.5)} \quad \liminf_{r \to R} \frac{w(r, F^{(\alpha, \beta)})}{r(R/R - r)^{\rho + 1}} \leq \rho t < \rho T \leq \limsup_{r \to R} \frac{w(r, F^{(\alpha, \beta)})}{r(R/R - r)^{\rho + 1}}.
\]

**Proof.** The proof proceeds exactly on the lines of Lemma 1, hence details are omitted.

**Definition.** A real valued GBASP \( F^{(\alpha, \beta)} \) is said to be of regular growth if \( 0 < \lambda = \rho < \infty \) and further, it is of perfectly regular growth if \( t = T \).

A real valued GBASP \( F^{(\alpha, \beta)} \) which are not of regular growth are called of irregular growth.

**Lemma 3.** The lower type of a real valued GBASP \( F^{(\alpha, \beta)} \) of irregular growth is zero.

**Proof.** If \( F^{(\alpha, \beta)} \) is of irregular growth than \( \rho > \lambda > 0 \). We have
\[
\liminf_{r \to R} \frac{\log^+ \log^+ M(r, F^{(\alpha, \beta)})}{\log (R/R - r)} = \lambda.
\]
Since \( M(r, F^{(\alpha, \beta)}) \to \infty \) as \( r \to R \), \( \log^+ \) may be replaced by \( \log \). For given \( \varepsilon > 0 \) and \( r > r_0 (\varepsilon) \),
\[
\text{(3.6)} \quad \log M(r, F^{(\alpha, \beta)}) < (R/R - r)^{\lambda - \varepsilon}.
\]
whereas for a sequence of values of $r \to \infty$,

\[(3.7) \quad \log M(r, F^{(\alpha, \beta)}) < (R/R - r)^{\lambda + \varepsilon}. \]

Dividing (3.6) and (3.7) by $(R/R - r)^\rho$ and passing to limit the argument shows that

\[\lim \inf_{r \to R} \frac{\log M(r, F^{(\alpha, \beta)})}{\log (R/R - r)} = 0.\]

From Lemma 3, we conclude that $t > 0$ is only limited to the study of a GBASP $F^{(\alpha, \beta)}$ of regular growth. In such case we define

\[\lim \inf_{r \to R} \frac{\log M(r, F^{(\alpha, \beta)})}{\log (R/R - r)^\lambda} = t_\lambda.\]

The quantity $t_\lambda$ is termed as $\lambda$-type of a real valued GBASP $F^{(\alpha, \beta)}$. It is significant to mention that there exist a GBASP $F^{(\alpha, \beta)}$ for which $t_\lambda$ is nonzero and finite. For such $F^{(\alpha, \beta)}$, we shall utilise the comparison function $F^{(\alpha, \beta)}$ analogous to proximate type as $\lambda$-proximate type $S_\lambda(r)$. The significance of $S_\lambda(r)$ is justified in Theorem 4.

**Theorem 3.** Let a real valued GBASP $F^{(\alpha, \beta)}$ regular in open hypersphere $\Sigma^{(\alpha, \beta)}$ having order $\rho$ ($0 < \rho < \infty$) and type $T$ ($0 \leq T \leq \infty$) such that limits in (3.2) and (3.5) exist. Then, for a positive real number $\eta$, 

\[\log (\eta^{-1} M(r, F^{(\alpha, \beta)})/(R/R - r)^\rho) \text{ is a proximate type of a GBASP } F^{(\alpha, \beta)}.\]

**Proof.** For a given constant $\eta$ ($0 < \eta < \infty$) let

\[(3.8) \quad S_\rho(r) = \frac{\log (\eta^{-1} M(r, F^{(\alpha, \beta)}))}{\log (R/R - r)^\rho}.\]

Since $\log M(r, F^{(\alpha, \beta)})$ is positive, continuous and increasing function of $r$ for $r > r_0 > 0$, which is differentiable in adjacent open intervals, it follows that $S_\rho(r)$ satisfies (iii). Existence of limit in (3.5) implies that $F^{(\alpha, \beta)}$ is of perfectly regular growth and moreover, $S_\rho(r) \to T$ as $r \to R$.

Differentiating (3.8), we get

\[\frac{S_\rho'(r)}{S_\rho(r)} = \frac{M(r, F^{(\alpha, \beta)})}{M(r, F^{(\alpha, \beta)}) \log (\eta^{-1} M(r, F^{(\alpha, \beta)}))} = \frac{\rho R}{R - r},\]
so that

\[(3.9) \quad \frac{(R-r)S'_\rho(r)}{RS_\rho(r)} = \frac{(R-r)w(r, F^{(\alpha, \beta)})}{Rr \log(\eta^{-1}M(r, F^{(\alpha, \beta)}))} - \rho.\]

Again, limits in (3.2) exist by assumption, hence

\[\frac{(R-r)S'_\rho(r)}{RS_\rho(r)} \to 0 \quad \text{as} \quad r \to R.\]

Thus \(S_\rho(r)\) satisfies the condition (ii).

From (3.8), (iv) is readily obtained. In this way all the assertions for \(S_\rho(r)\) to be a proximate type of GBASP \(F^{(\alpha, \beta)}\) are satisfied and hence the theorem.

**THEOREM 4.** Let a real valued GBASP \(F^{(\alpha, \beta)}\) regular in open hypersphere \(\Sigma^{(\alpha, \beta)}_R\) and having order \(\rho\), lower order \(\lambda\) \((0 < \lambda \leq \rho < \infty)\), type \(T\) and lower type \(t\). Then

\[(3.10) \quad \frac{c}{t} \leq \lim_{r \to R} \frac{(R-r)S'_\rho(r)}{RS_\rho(r)} + \rho \leq \frac{c}{t},\]

where

\[(3.11) \quad \lim_{r \to R} \sup_{r \to R} \frac{w(r, F^{(\alpha, \beta)})}{r(R/R-r)^{\rho+1}} = \frac{c}{d},\]

Moreover, if \(F^{(\alpha, \beta)}\) is of irregular growth then

\[(3.12) \quad -\infty \leq \lim_{r \to R} \frac{(R-r)S_\lambda(r)}{RS(r)} \leq \frac{d}{t_\lambda} - \lambda,\]

where \(S_\lambda(r)\) is a function in (3.8) corresponding to \(\lambda\) and \(t_\lambda\) is the \(\lambda\)-type of \(F^{(\alpha, \beta)}\).

**PROOF.** By (3.1) and the definition of type \(T\) and lower type \(t\) we observe that

\[(3.13) \quad \lim_{r \to R} \sup_{r \to R} \frac{1}{(R/R-r)^{\rho}} \int_{r_0}^r \frac{w(r, F^{(\alpha, \beta)})}{x} dx = \frac{T}{t},\]

Similarly, for GBASP \(F^{(\alpha, \beta)}\) of irregular growth,

\[(3.14) \quad \lim_{r \to R} \sup_{r \to R} \frac{1}{(R/R-r)^{\rho}} \int_{r_0}^r \frac{w(r, F^{(\alpha, \beta)})}{x} dx = t_\lambda.\]
Fix $r_0 \in (0, \infty)$ such that $\eta = \log M(r_0, F^{(\alpha, \beta)})$. Hence

$$\lim_{r \to r_0} \left( \frac{1}{\eta} M(r, F^{(\alpha, \beta)}) \right) = \int_{r_0}^r \frac{w(r, F^{(\alpha, \beta)})}{x} \, dx.$$ 

Dividing by $(R/R - r)^\rho$ and differentiating with respect to $r$, we get for almost all values of $r > r_0$,

$$\frac{S'_\rho(r)}{S_\rho(r)} = \frac{w(r, F^{(\alpha, \beta)})}{r \int_{r_0}^r \frac{w(x, F^{(\alpha, \beta)})}{x} \, dx} - \frac{\rho R}{R - r},$$

and this, on simplification, gives (3.9). Now, proceeding to limits in (3.9) and making use of (3.11) and (3.13), the inequalities in (3.10) follows at one.

In case $\rho > \lambda$, we have

$$S_\lambda(r) = \frac{\log(\eta^{-1} M(r, F^{(\alpha, \beta)})}{(R/R - r)\lambda} = \frac{1}{(R/R - r)\lambda} \int_{r_0}^r \frac{w(r, F^{(\alpha, \beta)})}{x} \, dx.$$ 

By the parallel arguments and making use of (3.14), (3.12) can be disposed of.

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