COMPUTATION OF MATRIX SPLITTINGS
AND THEIR APPLICATIONS

ABSTRACT: A few methods for constructing the index splitting and proper splitting are presented. Also, corresponding representations of the Drazin inverse and the Moore-Penrose inverse are introduced. In partial cases we get known results from [3], [4], [12] and [13]. We also give some convergence criteria for the iterative method for computing the minimal $P$-norm solution of a given singular linear system $Ax = b$, introduced in [12].

KEY WORDS: index splitting, proper splitting, Drazin inverse, Moore-Penrose inverse, block representation.

1. INTRODUCTION

Let $C^{m\times n}$ be the set of $m \times n$ complex matrices, and $C^{m\times n}_r = \{X \in C^{m\times n} : \operatorname{rank}(X) = r\}$. The first $r$ columns of $A$ and the first $r$ rows of $A$ we denote by $A_l^r$ and $A_r$, respectively. Similarly, $A_r^l$ and $A_r$ denote the last $r$ columns of $A$ and the last $r$ rows of $A$, respectively. By $I_r$ we denote the identity matrix of the order $r$, and $O$ denotes an appropriate zero block. We use $\mathcal{N}(A)$ to denote the kernel and $\mathcal{R}(A)$ to denote the image of $A$, and $\rho(A)$ to denote the spectral radius of $A$. The index of a square matrix $A$ is denoted by $\operatorname{ind}(A)$.

For any matrix $A \in C^{m\times n}$ consider the following equations in $X$:

(1) $AXA = A,$
(2) $XAX = X,$
(3) $(AX)^* = AX,$
(4) $(XA)^* = XA$

where the superscript * denotes conjugate and transpose matrix. Also, in the case $m = n$, consider the following equations:

(5) $AX = XA$

for a positive integer $k = \operatorname{ind}(A) = \min\{p : \operatorname{rank}(A^{p+1}) = \operatorname{rank}(A^p)\}$. For a sequence $S$ of the elements from the set $\{1, 2, 3, 4, 5\}$, the set of matrices obeying the equations represented in $S$ is denoted by $A\{S\}$. A matrix from $A\{S\}$ is called an $S$-inverse of $A$ and denoted by $A^{(S)}$. If $X$ satisfies the system of equations (1), (2) it is said to be a reflexive $g$-inverse of $A$, whereas the
Moore-Penrose inverse $X = A^\dagger$ of $A$ satisfies the set of the equations (1)-(4). A matrix $X = A^D$ is said to be the Drazin inverse of $A$ if the equations (1$^k$), (2) and (5) are satisfied. The group inverse $A^\#$ is the unique $\{1, 2, 5\}$ inverse of $A$, and exists if and only if $\text{ind}(A) = 1$.

Assume that the matrices $R, G$ are regular, $E, F$ are permutation matrices and $U, V$ are unitary matrices. Main block decompositions of a given matrix $M \in \mathbb{C}^{m \times n}$ are used in [2], [15]:

$$(T_1) \quad M = R^{-1}\begin{bmatrix} B & O \\ O & O \end{bmatrix}G^{-1} = R^{-1}\begin{bmatrix} I_r \\ O \end{bmatrix}B[I_r, O]G^{-1};$$

$$(T_2) \quad M = U^*\begin{bmatrix} B & O \\ O & O \end{bmatrix}V^* = U^*\begin{bmatrix} I_r \\ O \end{bmatrix}B[I_r, O]V^*;$$

$$(T_3) \quad M = E^*\begin{bmatrix} A_{11} & A_{11}^T \\ S\overrightarrow{A}_{11} & S\overrightarrow{A}_{11}^T \end{bmatrix}F^* = E^*\begin{bmatrix} I_r \\ S \end{bmatrix}A_{11}[I_r, T]F^*.$$

In the papers [1], [2], [7], [9], [10], [11] there are represented various classes of generalized inverses in terms of block decompositions. In this paper we present a few alternative representations, based on the index splitting and block factorizations $(T_1) - (T_3)$.

We restate the following definition of the index splitting from [12]:

**DEFINITION 1.1.** Let $A \in \mathbb{C}^{m \times n}$ with $\text{ind}(A) = k$. Then the splitting $A = U - V$ is called an index splitting provided that

$$\mathcal{R}(U) = \mathcal{R}(A^k), \quad \mathcal{N}(U) = \mathcal{N}(A^k).$$

In the case $k = 1$ the index splitting reduces to a proper splitting, which is investigated in [3].

The matrix splittings and their applications was investigated by numerous authors [3], [4], [12], [13], [14]. The matrix splittings are useful in representation of generalized inverses as well as in the computation of various iterative solutions of the linear system $Ax = b$. These iterative solutions are based on the general iterative formula $x_{i+1} = x_i + U^-Vx_i + U^-b$, where $U^-$ is a kind of generalized inverse of $U$ [13]. The nonsingular case was investigated in [14]. The proper splitting and its applications in computation of the Moore-Penrose inverse and the best approximate solution of the linear system was introduced in [3] and [4]. The index splitting, which is used in the representation of the Drazin inverse and the minimal $P$-norm solution was investigated in [12] and [13].
An effective method for constructing the proper splitting was given in [3]. For the construction of the matrix $U$ it is necessary to find a factorization $(T_3)$ for the matrix $M = A$, and then replace the matrix $A_{11}$ by an arbitrary (easily) invertible matrix $U_{11}$ of the order $r$. Similar method for constructing the index splitting was introduced in [12]. This method is based on the block decomposition $(T_1)$ of the matrix $M = A^k$, $k = \text{ind}(A)$, and also replaces $A_{11}$ by arbitrary nonsingular matrix $U_{11}$ of the order $r$. In this paper we are motivated by the idea that it is possible to replace the matrices $B$ in block decompositions $(T_1)$ and $(T_2)$ by an arbitrary invertible matrix $H_u$.

In the second section we give two methods for construction of the index splitting and the proper splitting. Using these constructions of the matrix splitting we introduce a few methods for computation of the Drazin inverse and the Moore-Penrose inverse. In the third section we introduce several convergence criteria for the iterative method, introduced in [12], which can be used in computation of the minimal $P$-norm solution of a given singular linear system $Ax = b$, $b \in \mathcal{R}(A^k)$, $k = \text{ind}(A) = 1$.

2. CONSTRUCTION OF THE INDEX SPLITTING AND PROPER SPLITTING

**Theorem 2.1.** Let $A \in \mathbb{C}^{n \times n}$ be a square matrix satisfying $\text{ind}(A) = k$ and $r = \text{rank}(A^k)$. Let $H_u$ be an arbitrary invertible $r \times r$ matrix, and let the matrices $R$, $G$, $E$, $F$, $U$ and $V$ are determined by the application of the corresponding block decompositions $(T_i)$ on the matrix $M = A^k$. Then the splitting $A = U_A V_A$ is the index splitting of $A$ if and only if $U_A$ and $V_A$ are defined in one of following expressions $(S_i)$, where $(S_i)$ corresponds to the block decomposition $(T_i)$ of the matrix $A^k$, $i \in \{1, 2, 3\}$. Also, the Drazin inverse of $A$ is equal to $A^D = (I - U_A^r V_A) V_A^{-1} U_A^r$, where $U_A^r$ is defined in $(S_i)$, $i \in \{1, 2, 3\}$.

\begin{align*}
(S_1) \quad U_A &= (R^{-1})^r H_u (G^{-1})_{rl} = R^{-1} \begin{bmatrix} H_u & 0 \\ 0 & 0 \end{bmatrix} G^{-1}, \\
U_A^r &= (R^{-1})^r (H_u (G^{-1})_{rl}) (R^{-1})^r H_u (G^{-1})_{rl}, \\
(S_2) \quad U_A &= (U^*)^r H_u (V^*)_{rl} = U^* \begin{bmatrix} H_u & 0 \\ 0 & 0 \end{bmatrix} V^*, \\
U_A^r &= (U^*)^r (H_u (V^*)_{rl}) (U^*)^r H_u (V^*)_{rl},
\end{align*}
\((S_3)\) \[ U_A = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} H_u[I_r, T] F^* = E^* \begin{bmatrix} H_u & H_u T \\ S H_u & S H_u T \end{bmatrix} V^*, \]

\[ U_A^# = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} \left( H_u[I_r, T](EF)^* \begin{bmatrix} I_r \\ S \end{bmatrix} \right)^{-2} H_u[I_r, T] F^* = \]

\[ = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} ((EF)^* + T(EF)^* S)^{-1} H_u^{-1} ((EF)^* + T(EF)^* S)^{-1}[I_r, T] F^*. \]

**Proof.** \((S_1)\) Assume that \(A = U_A - V_A\) is the index splitting of \(A\). Since \(\mathcal{R}(U_A) = \mathcal{R}(A^k)\) and \(\mathcal{N}(U_A) = \mathcal{N}(A^k)\), there exist nonsingular matrices \(P\) and \(Q\) such that \(U_A = PA^k = A^k Q\) (see [3]). Let the matrices \(P\) and \(Q\) are partitioned as

\[ P = R^{-1} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} R, \quad Q = G \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} G^{-1}. \]

Using block decomposition \((T_i)\) of the matrix \(A^k\), one can verify the following:

\[ U_A = R^{-1} \begin{bmatrix} P_{11} B & 0 \\ P_{21} B & 0 \end{bmatrix} G^{-1} = R^{-1} \begin{bmatrix} B Q_{11} & B Q_{12} \\ 0 & 0 \end{bmatrix} G^{-1}. \]

A solution of this matrix equation is given by

\[ P_{11} = H_u B^{-1}, \quad Q_{11} = B^{-1} H_u, \quad P_{21} = Q_{12} = 0. \]

Hence,

\[ U_A = R^{-1} \begin{bmatrix} H_u & 0 \\ 0 & 0 \end{bmatrix} G^{-1} = (R^{-1})^T H_u (G^{-1})^T. \]

On the other hand, assume that \(U_A\) is defined as in part \((S_1)\) of theorem and \(V_A = U_A - A\). Then, for the nonsingular matrices

\[ P = R^{-1} \begin{bmatrix} H_u B^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} R, \quad Q = G \begin{bmatrix} B^{-1} H_u & 0 \\ Q_{12} & Q_{22} \end{bmatrix} G^{-1} \]

we have \(U_A = PA^k = A^k Q\). Hence, \(\mathcal{R}(U_A) = \mathcal{R}(A^k)\), \(\mathcal{N}(U_A) = \mathcal{N}(A^k)\), and \(A = U_A - V_A\) is the index splitting of \(A\).
Also, \( U_A = PQ \), where \( P = (R^{-1})^r \), \( Q = H_u(G^{-1})_r \), is a full-rank factorization of \( U_A \). According to Cline’s general representation of the group inverse [5], we get
\[
U_A^* = P(QP)^{-2}Q = (R^{-1})^r(H_u(G^{-1})_r(R^{-1})^r)^{-2}H_u(G^{-1})_r.
\]

\((S_2)\) Assume that \( A = U_A - V_A \) is the index splitting of \( A \) and the nonsingular matrices \( P \) and \( Q \) are partitioned as
\[
P = U^*[P_{11} P_{12} P_{21} P_{22}] U, \quad Q = V[Q_{11} Q_{12} Q_{21} Q_{22}] V^*.
\]
Using \( U_A = PA^k = A^kQ \) we obtain
\[
U_A = U^* [P_{11}B O P_{21}B O] V^* = U^* [BQ_{11} BQ_{12} O O] V^*.
\]
Consequently, we have
\[
P_{11}B = BQ_{11}, \quad P_{21} = O, \quad Q_{12} = O.
\]
For example, we can use
\[
Q_{11} = B^{-1}P_{11}B, \quad P_{21} = O, \quad Q_{12} = O, \quad P_{11}, P_{12}, P_{22}, Q_{21}, Q_{22} \text{ arbitrary.}
\]
In this case is
\[
U_A = U^* [I_r O P_{11}B[I_r K] V^* = (U^*)^r H_u(V^*)_r,
\]
where \( H_u = P_{11}B \).

On the other hand, assume that \( U_A \) is presented in \((S_2)\) and \( V_A = A - U_A \).
For nonsingular matrices
\[
P = U^*[H_u B^{-1} O P_{12} P_{22}] U, \quad Q = V[B^{-1}H_u O Q_{12} Q_{22}] V^*.
\]
we have \( U_A = PA^k = A^kQ \). This \( \mathcal{R}(U_A) = \mathcal{R}(A^k) \), \( \mathcal{N}(U_A) = \mathcal{N}(A^k) \), and \( A = U_A - V_A \) is the index splitting of \( A \).

Moreover, \( U_A = PQ \), where \( P = (U^*)^r \), \( Q = H_u(V^*)_r \), is a full-rank factorization of \( U_A \). Again, using the Cline’s general representation of the group inverse from [5], we get
\[
U_A^* = P(QP)^{-2}Q = (U^*)^r(H_u(V^*)_r(U^*)^r)^{-2}H_u(V^*)_r.
\]
\((S_3)\) This part of the proof is known from [12] and [13].
**Remark 2.1.** In the case \( \text{ind}(A) = 1 \), from Theorem 2.1, we get analogous representations of the group inverse.

Applying the block decompositions \((T_1) - (T_3)\) on the matrix \(A\) we get analogous results for the proper splitting and the Moore-Penrose inverse.

**Theorem 2.2.** Let \(A \in \mathbb{C}^{m \times n}\) be an arbitrary \(m \times n\) matrix of rank \(r\). Let \(H_u\) be an invertible \(r \times r\) matrix, and let the matrices \(R, G, E, F, U\) and \(V\) are determined by the corresponding block decompositions \((T_1) - (T_3)\) of the matrix \(M = A\). The splitting \(A = U_A - V_A\) is the proper splitting of \(A\) if and only if \(U_A\) is defined in one of the expressions \((S_i)\), corresponding to the block decompositions \((T_i)\) of the matrix \(A\), \(i = 1, 2, 3\).

Also, the Moore-Penrose inverse \(A^+\) of \(A\) is equal to \(A^+ = (I - U_A^*V_A)^{-1}U_A^+\), where \(U_A\) is defined in the part \((S_i)\) of Theorem 2.1 and \(U_A^+\) is equal to the corresponding between the following expressions \((M_i)\), where \((M_i)\) arises from the block decomposition \((T_i)\), \(i \in \{1, 2, 3\}\) of the matrix \(A\).

\[
(M_1) \quad U_A^* = (H_u(G^{-1})_n^\top)^*(((R^{-1})_n^\top)^*A(H_u(G^{-1})_n^\top)^*)^{-1}((R^{-1})_n^\top)^*
\]

\[
(M_2) \quad U_A^+ = V_n^\top (U_n^\top A V_n^\top)^{-1}U_n^\top
\]

\[
(M_3) \quad U_A^+ = F \begin{bmatrix} I_r^\top & I_r^\top \\ T^* & S^* \end{bmatrix} E A F \begin{bmatrix} I_r^\top & I_r^\top \\ T^* & S^* \end{bmatrix}^{-1} [I_r, S^*] E.
\]

**Proof.** The representation \(A^+ = (I - U_A^*V_A)^{-1}U_A^+\) is known from [4]. The results \((M_1) - (M_3)\) follows from the known representation of the Moore-Penrose inverse [2]: if \(U_A = PQ\) is a full-rank factorization of \(U_A\), then \(U_A^+ = Q^* (P^* A Q^*)^{-1} P^*\).

### 3. Computing Minimal P-norm Solution

In this section we investigate the iterative method for the minimal P-norm solution of the following singular linear system

\[
(3.1) \quad Ax = b, \quad b \in \mathcal{R}(A^k), \quad k = \text{ind}(A) = 1.
\]
Our investigations are based on the application of the block decompositions \((T_1)\) and \((T_2)\). A method which is based on the block decomposition \((T_3)\) is introduced in [12].

**Theorem 3.1.** Consider the singular linear system \((3.1)\). Assume that the matrices \(R, G, E, F, U\) and \(V\) are determined by the corresponding block decompositions \((T_1) - (T_3)\) of the matrix \(A\). Let \(H_u\) be an appropriate invertible \(r \times r\) matrix. Assume that \(A = U_A - V_A\) is an index splitting of \(A\), and \(b \in \mathbb{C}^n\) is an arbitrary vector. Then the following iterative process from [12]:

\[
(3.2) \quad x_{i+1} = U_A^b V_A x_i + U_A^b b
\]

converges to \(A^0 b\), for every \(x_0 \in \mathbb{C}^n\), if and only if one of the following conditions \((K_i)\) is satisfied, where \((K_i)\) arises from the block decomposition \((T_i)\), \(i \in \{1, 2, 3\}\):

\((K_1)\) The eigenvalues of the matrix \(H_u^{-1} B\) are greater than 0 and smaller than 2.

\((K_2)\) The eigenvalues of \(H_u^{-1} A_{11}\) are greater than 0 and smaller than 2.

**Proof.** In the cases \((K_1)\) and \((K_2)\) it is not difficult to verify

\[
(3.3) \quad \rho(U_A^b V_A) = \rho(I - H_u^{-1} B).
\]

For example, the part \((K_2)\) can be verified as follows. Using the part \((S_1)\) of Theorem 2.1 and the block decomposition \((T_2)\) of the matrix \(A\), we get

\[
U_A^b V_A = (R^{-1})^n (H_u (G^{-1})_{\mathcal{R}} (R^{-1})_{\mathcal{R}})^{-2} H_u (G^{-1})_{\mathcal{R}} (R^{-1})_{\mathcal{R}} (H_u - B) (G^{-1})_{\mathcal{R}} =
\]

\[
= (R^{-1})^n ((G^{-1})_{\mathcal{R}} (R^{-1})_{\mathcal{R}})^{-1} (I - H_u^{-1} B) (G^{-1})_{\mathcal{R}}.
\]

Now, using

\[
(G^{-1})_{\mathcal{R}} (R^{-1})_{\mathcal{R}} ((G^{-1})_{\mathcal{R}} (R^{-1})_{\mathcal{R}})^{-1} = I,
\]

according to Theorem 2 from [3], we get (3.3). Using the known result from [12], iterative method (3.2) converges if and only if

\[
(3.4) \quad \rho(U_A^b V_A) < 1.
\]

This part of the proof can be completed using (3.4) and (3.3).
Similarly, in the case \((K_3)\) we get
\[
\rho(U_A^* V_A) = \rho(I - H_u^{-1} A_{11}),
\]
and the proof can be completed using the known principle.

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Received on 21.02.2000 and, in revised form, on 09.11.2000.