SOME REMARKS ON ALMOST PERIODIC FUNCTIONS

ABSTRACT: In this paper we present the definition and some properties of $(IC)$-a.p. functions, i.e. uniformly almost periodic $(B$-a.p.) functions with their indefinite integrals. Next, we give the definition and some properties of $(IC)^{(n)}$-a.p. functions, i.e. uniformly almost periodic functions with their $n$ derivatives and indefinite integrals, and $(IC)^{(n)}_{\omega}$-a.p. functions, i.e. uniformly almost periodic functions with their every derivatives, with respect to a positive sequence $a = (a_i)$, and indefinite integrals.

KEY WORDS: uniformly almost periodic function, derivative of order n, indefinite integral.

1. $(IC)$ – ALMOST PERIODIC FUNCTIONS

1.1. Definitions

We first give basic notations related to uniformly almost periodic functions with their indefinite integrals.

By $C(R)$ we denote the set of functions from $R$ into itself which are continuous. Denote for $f, g \in C(R)$

$$T(f_a, g_b)(u) = |F_a(u) - G_b(u)| \quad \text{for} \quad u \in R,$$

where $a, b \in R$, $f_a(x) \equiv f(x + a)$, $g_b(x) \equiv g(x + b)$, $F_a(x) \equiv F(x + a) = \int_0^{x+a} f(s) \, ds$, $G_b(x) \equiv G(x + b) = \int_0^{x+b} g(s) \, ds$, and in the follow we define the $(ID)$ – distans, putting

$$(ID)(f, g) = \sup_{t \in R} (|f(t) - g(t)| + T(f, g)(t)).$$

We say that an $f \in C(R)$ is $(IC)$ – bounded iff $(ID)(f) < \infty$, where $(ID)(f) = (ID)(f, 0)$. Let $f_h(x) \equiv f(x + h)$. We say that $f \in C(R)$ is an $(IC)$-continuous function iff $\lim_{h \to 0} (ID)(f, f_h) = 0$. A sequence $(f_k)$ in $C(R)$ will be called $(ID)$-convergent to an $f \in C(R)$ iff $\lim_{k \to \infty} (ID)(f, f_k) = 0$.

THEOREM 1.1. If a sequence $(f_k)$ of $(IC)$-continuous functions is $(ID)$-convergent to a function $f \in C(R)$, then an $f$ is $(IC)$-continuous.
**Proof.** Since a sequence \((f_k)\) is \((ID)\)-convergent to an \(f \in C(R)\), it follows that for an arbitrary \(\varepsilon > 0\) there exists a \(k_0 > 0\) such that
\[
(ID)(f, f_{k_0}) \leq \frac{\varepsilon}{3}.
\]
By \((IC)\)-continuity of \(f_{k_0}\), there exists a \(\delta = \delta(\varepsilon, k_0) > 0\) such that
\[
(ID)((f_{k_0})_h, f_{k_0}) \leq \frac{\varepsilon}{3} \quad \text{for} \quad h \in R, \quad |h| < \delta,
\]
where \((f_{k_0})_h(x) = f_{k_0}(x + h)\). Hence for \(|h| < \delta\) we obtain
\[
(ID)(f, f_h) \leq (ID)(f, f_{k_0}) + (ID)(f_{k_0}, (f_{k_0})_h) + (ID)((f_{k_0})_h, f_h) \leq \varepsilon,
\]
because \(T((f_{k_0})_h, f_h) = |(F_{k_0})_h - F_h|\), and so \((ID)((f_{k_0})_h, f_h) = (ID)(f, f_{k_0})\).
Thus \(an \ f\) is \((IC)\)-continuous.

A set \(E \subset R\) is called *relatively dense* iff there exists a positive number \(l\) such that in every open interval \((\alpha, \alpha + l)\), \(\alpha \in R\), there is at least one element of the set \(E\). A number \(\tau \in R\) is called an \((ID), \varepsilon) – \text{almost period} (\((ID), \varepsilon) – \text{a.p.})\) of a function \(f \in C(R)\) iff \((ID)(f, f_\tau) \leq \varepsilon\), for \(\varepsilon > 0\). Let \((IE)\{\varepsilon; f\}\) denote the set of \((ID), \varepsilon) – \text{a.p.}\) periods of \(an \ f\).

A function \(f \in C(R)\) is called \((IC)\)-*almost periodic* ((IC)-a.p.) iff for each \(\varepsilon > 0\) the set \((IE)\{\varepsilon; f\}\) is relatively dense. By \((IC)\) we denote the set of (IC)-a.p. functions.

It is obvious that every (IC)-a.p. function is uniformly almost periodic (B-a.p.)

As regards the relation between \((IC)\)-boundedness and \((IC)\)-continuity, there holds the following:

**Remark 1.2.** For (IC)-a.p. functions classes of (IC)-bounded functions and of (IC)-continuous functions are identical. Generally, the class of (IC)-bounded functions and the class of (IC)-continuous functions are different. More, there exist functions belonging to one of these classes and not belonging to the other one (see Examples:1.16,1.17).

Denote
\[
F(u) = \int_0^u f(s) \, ds \quad \text{for} \quad u \in R.
\]
There are known the following:
**Lemma 1.3.** Let \( f \) be a B-a.p. function. Then for an arbitrary \( \varepsilon > 0 \) there exists an \( \varepsilon = \varepsilon(\varepsilon) > 0 \) such that \( \varepsilon < \varepsilon/3 \) and every \( \varepsilon \)-a.p. of an \( f \) is \((\varepsilon/3)\)-a.p. of the bounded indefinite integral \( F \) (see [5], p.29).

**Lemma 1.4.** For any B-a.p. functions \( f, g \) and for an arbitrary \( \varepsilon > 0 \) there exists a relatively dense set of their commonly \( \varepsilon \)-a.periods (see [5], p. 203 or [4], p. 432).

**Remark 1.5.** A function \( f \) is \((IC)\)-a.p. if and only if \( f \) and its indefinite integral \( F \) are uniformly a.p. functions.

**Substantiation (Necessity)** It is easily seen that if \( f \) is a \((IC)\)-a.p. function, then functions \( f \) and \( F \) are B-a.p.

**Sufficiency** Let \( \varepsilon > 0 \). Since \( f, F \) are B-a.p. functions, so using Lemma 1.4 we obtain that there exists the relatively dense set \( E\{\varepsilon/2; f, F\} \) of their commonly \( \varepsilon \)-a.periods. Thus for \( \tau \in E\{\varepsilon/2; f, F\} \) we have the estimation
\[
(ID)(f, f_\tau) \leq \varepsilon.
\]
We conclude, \( E\{\varepsilon/2; f, F\} \subset (IE)\{\varepsilon; f\}, \) so \( f \in (IC) \).

Moreover, using the Bohl-Bohr Theorem on the bounded indefinite integral of a uniformly a.p. function (see [5], we have:

**Remark 1.6.** A function \( f \) is \((IC)\)-a.p. if and only if an \( f \) is a B-a.p. function and its indefinite integral \( F \) is bounded.

**Remark 1.7.** The set of values of a B-a.p. function is connected.

**1.2. Basic Properties**

**Theorem 1.8.** If \( f \) is an \((IC)\)-a.p. function, then:

(i) \( an \ f \) is \((IC)\)-bounded,

(ii) \( an \ f \) is \((IC)\)-continuous.

**Proof.** Let \( f \in (IC) \).

(i) According to Remark 1.5, functions \( f \) and \( F \) are bounded, as uniformly a.p. functions. Thus we obtain \((ID)(f) \leq M, \) where \( M > 0 \) is a constant.

(ii) Similarly, by Remark 1.5, functions \( f \) and \( F \) are uniformly continuous (see [5], p. 22). Consequently, an \( f \) is \((IC)\)-continuous.
Now, we shall be occupied with $(IC)$-aperiodicity of a linear combination of $(IC)$-a.p. functions, and next with $(IC)$-aperiodicity of a product of above functions.

**Theorem 1.9.** The following statements hold:

(i) A linear combination of $(IC)$-a.p. functions is an $(IC)$-a.p. function.

(ii) A product of two $B$-a.p. functions is $(IC)$-a.p. if and only if the indefinite integral of a product of these functions is bounded.

**Proof.** (i) Let $f, g \in (IC)$. By Remark 1.5 and theorem on a sum of uniformly a.p. functions (see [5], p. 27), immediately we obtain $f + g \in (IC)$. Moreover, $cf \in (IC)$, where $c$ is a constant.

(ii) Let $f, g$ are $B$-a.p. functions.

**(Necessity)** Since $fg \in (IC)$, so its indefinite integral $F_{fg}$ is $B$-a.p., where $F_{fg}(u) = \int_0^u f(s)g(s)ds$ for $u \in R$. Thus $F_{fg}$ is bounded (see [5], p. 22).

**(Sufficiency)** Since $fg$ is a uniformly a.p. function and its indefinite integral $F_{fg}$ is bounded, so, by Remark 1.6, we get that a product $fg$ is a $(IC)$-a.p. function.

**Corollary 1.10.** A product of two $(IC)$-a.p. functions is $(IC)$-a.p. if and only if the indefinite integral of a product of these functions is bounded.

**Theorem 1.11.** If a sequence $(f_k)$ of $(IC)$-a.p. functions is $(ID)$-convergent to a function $f \in C(R)$, then an $f$ is $(IC)$-a.p.

**Proof.** Since a sequence $(f_k)$ is $(ID)$-convergent to an $f \in C(R)$, it follows that for an arbitrary $\varepsilon > 0$ there exists a $k_0 > 0$ such that $(ID)(f, f_{k_0}) < \varepsilon/3$. We have $f_{k_0} \in (IC)$, so for $\tau \in (IE)\{\varepsilon/3; f_{k_0}\}$ there holds

$$(ID)(f, f_{\tau}) \leq (ID)(f, f_{k_0}) + (ID)(f_{k_0}, (f_{k_0})_\tau) + (ID)((f_{k_0})_\tau, f_{\tau}) \leq \varepsilon.$$  

Thus $(IE)\{\varepsilon/3; f_{k_0}\} \subset (IE)\{\varepsilon; f\}$, i.e. $f \in (IC)$.

In the following, we shall seek for sufficient condition under which the derivative $f'$ of a function $f \in (IC)$ will be an $(IC)$-a.p. function, as well.
THEOREM 1.12. If the derivative $f'$ of a B-a.p. function $f$ is uniformly continuous, then $f'$ is an (IC)-a.p. function.

PROOF. It is known that for a B-a.p. function $f$ such that $f'$ is uniformly continuous we have that $f'$ is also B-a.p. (see [5], p. 28). Moreover, it follows

$$F_{f'}(t) = f(t) - f(0) \quad \text{for} \quad t \in \mathbb{R},$$

where $F_{f'}(u) = \int_0^u f'(s)ds$, $u \in \mathbb{R}$. Therefore $F_{f'}$ is a uniformly a.p. function. Consequently, according to Remark 1.5, we obtain $f' \in \overline{\text{(IC)}}$.

Now, we shall investigate the indefinite integral of a uniformly a.p. function. It is known that if the indefinite integral of a B-a.p. function is bounded, then this integral is a B-a.p. function (see [5], the Bohl-Bohr Theorem). We shall give more, namely there follow:

THEOREM 1.13. If $f$ is a B-a.p. function and its indefinite integral $F$ is (IC)-bounded, then $F$ is an (IC)-a.p. function and a $C^{(1)}$ – a.p. function (see [1]).

PROOF. By theorem on the bounded indefinite integral of a uniformly a.p. function, an $F$ is B-a.p., as a bounded function. Moreover, similarly we have $F_{f'}$ is a B-a.p. function, where $F_{f'}(u) = \int_0^u F(s)ds$ for $u \in \mathbb{R}$, because $F_{f'}$ is bounded too. We conclude, using Remark 1.5, that $F \in \overline{\text{(IC)}}$. According to Theorem 7 in [1], we obtain $F \in \overline{C^{(1)}}$. The proof is complete.

REMARK 1.14. If $f$ is an (IC)-a.p. function and the indefinite integral $F_{f'}$ of a function $F$ is bounded, then the indefinite integral $F$ of an $f'$ is (IC)-a.p. and $C^{(1)}$ – a.p. (see [1]).

Finally, we shall be occupied with (IC)-a-periodicity of a function $f$.

THEOREM 1.15. The following statements hold:

(i) If $f$ is a uniformly continuous function and its indefinite integral $F$ is uniformly a.p., then an $f$ is (IC)-a.p.

(ii) If the derivative $f'$ is a uniformly a.p. function and a function $f$ is (IC)-bounded, then an $f$ is (IC)-a.p. and $C^{(1)}$ – a.p. (see [1]).

(iii) If the derivative $f'$ is an (IC)-a.p. function and the indefinite integral $F$ of a function $f$ is bounded, then an $f$ is (IC)-a.p. and $C^{(1)}$ – a.p. (see [1]).
1.3. Examples

First, we shall give examples of: an \((IC)\)-bounded function which is not \((IC)\)-continuous (Example 1.16) and an \((IC)\)-continuous function which is not \((IC)\)-bounded (Example 1.17).

Let \((\mathbb{Q})\) denote the set of irrational numbers.

**Example 1.16.** Let \(f\) be the function defined by \(f(x) = \sin \varphi(|x|)\) for \(x \in \mathbb{R}\), where \(\varphi\) is a \(\varphi\)-function (see [6] or [7]) strictly increasing such that the inverse function \((\varphi^{-1})\) has the finite derivative \((\varphi^{-1})'\) on \((0, \infty)\), satisfying the following condition:

\[
(\varphi^{-1})'(t) \downarrow 0 \quad \text{with} \quad t \to \infty.
\]

Then the function \(f\) is \((IC)\)-bounded, because in paper [1], p. 4, there shows that the indefinite integral \(F(x) = \int_0^x \sin \varphi(|s|) \, ds, \quad x \in \mathbb{R}\), is bounded. Moreover, \(f\) is a bounded function, as well. However, the function \(f\) is not \((IC)\)-continuous, since, by the condition (1), \(f\) is not a uniformly continuous function.

In particular, we may also take \(\varphi(u) = u^p\), with \(p > 1\), or \(\varphi(u) = e^u - 1\) for \(u \geq 0\).

**Example 1.17.** The function defined by \(f(x) = 1 + \cos x, \quad x \in \mathbb{R}\), has the unbounded indefinite integral \(F\), so is not \((IC)\)-bounded. However, \(f\) is an \((IC)\)-continuous function.

**Example 1.18.** The function \(f\) in the form \(f(x) = 2x \cos(\sqrt{x})\), \(x \in \mathbb{R}\), is unbounded, but its indefinite integral \(F(x) = \sin(x^2), \quad x \in \mathbb{R}\), is bounded.

It is easily given an example of an \((IC)\)-a.p. function. However, let us remark that every constant function which is different from zero doesn't belong to the class \((IC)\).

As regards the relation between \((IC)\)-a.p. and \(B\)-a.p. functions, there holds the following contain: \((IC) \subseteq \mathcal{B}\), where \(\mathcal{B}\) denote the set of uniformly a.p. functions (Examples: 1.19, 1.20).

**Example 1.19.** Let \(f\) be a \(B\)-a.p. function which has the bounded indefinite integral \(F\). Then \(f \in (IC)\).

In particular, we may also take the function \(f\) in the form \(f(x) = \sin x + \sin(\alpha x)\) for \(x \in \mathbb{R}\), where \(\alpha \in \mathbb{Q}\), since \(F\) is bounded.
**Example 1.20.** Let \( f \) be a B-a.p. function which has the unbounded indefinite integral \( F \). Then \( f \in (IC) \).

In particular, we may also take the function \( f \) defined by
\[
f(x) = 2 \cos x + \cos(\alpha x)
\]
for \( x \in R \), where \( \alpha \in NQ \). Then \( f \) is a uniformly a.p. function and \( f \not\in (IC) \).

Let us still remark that above function \( f \) is \( C^{(1)} \) – a.p. (see [1]), because the derivative \( f' \) is B-a.p.

Now, we shall give an example of an (\( IC \))-a.p. function which is not \( C^{(1)} \) – a.p. (Example 1.21 and see [1]). An example of a \( C^{(1)} \) – a.p. function which is not (\( IC \))-a.p. is the function from Example 1.20.

**Example 1.20.** Let
\[
f(x) = \begin{cases} 
\left( \frac{x - 2k}{\pi} \right)^2 \frac{1}{x - 2k} & \text{for } x \in \left( \frac{2k - 1}{\pi}, \frac{2k + 1}{\pi} \right) \setminus \left\{ \frac{2k}{\pi} \right\}, \\
0 & \text{for } x = \frac{2k}{\pi},
\end{cases}
\]
\[
g(x) = \begin{cases} 
\left( \frac{\sqrt{2x} - 2k}{\pi} \right)^2 \frac{1}{\sqrt{2x} - 2k} & \text{for } x \in \left( \frac{\sqrt{2(2k - 1)}}{2\pi}, \frac{\sqrt{2(2k + 1)}}{2\pi} \right) \setminus \left\{ \frac{\sqrt{2k}}{\pi} \right\}, \\
0 & \text{for } x = \frac{\sqrt{2k}}{\pi},
\end{cases}
\]
where \( k = 0, \pm 1, \pm 2, \ldots \).

Functions \( f \) and \( g \) are periodic with periods \( T_f = 2/\pi, \ T_g = \sqrt{2}/\pi \), respectively. Moreover, \( f \) and \( g \) are continuous functions on \( R \). Derivatives \( f', g' \) exist at every point \( x \in R \), but the derivative \( f' \) is not continuous at points \( x = 2k/\pi \) and the derivative \( g' \) is not continuous at points \( x = \sqrt{2k}/\pi, \ k = 0, \pm 1, \pm 2, \ldots \). Thus the sum \( h = f + g \) is a B-a.p. function and the derivative \( h' \in C(R) \). From where we obtain \( h \notin C^{(1)} \). Using Remark 1.6, we need only to see that the indefinite integral \( H(x) = \int_0^x h(s) ds, \ x \in R \), is bounded. Namely, there exist positive constants \( M_1 \) and \( M_2 \) such that for each \( x \in R \) there exists a \( k_0 \in Z = \{ \ldots , -2 , -1 , 0 , 1 , 2 , \ldots \} \) such that
\[ \left| \int_{0}^{x} f(s)ds \right| \leq \left| \int_{0}^{\pi} f(s)ds \right| + \left| \int_{(2k_\alpha+1)/\pi}^{x} f(s)ds \right| \leq M_1 \]

and

\[ \left| \int_{0}^{x} g(s)ds \right| \leq M_2. \]

Consequently, taking \( 0 < M = 2 \max \{M_1, M_2\} \), for every \( x \in R \) it follows \( |H(x)| \leq M \). Finally, we have \( h \in \overline{(IC)} \).

**Example 1.22.** Let \( f \) be a \( B \)-a.p. function which have the uniformly continuous derivative \( f' \) and the bounded indefinite integral \( F \). Then there holds \( f \in \overline{(IC)} \cap \overline{C^{(1)}} \).

In particular, let \( f \) be the function in the form \( f(x) = \cos(\alpha x) + \cos(\beta x) \) for \( x \in R \), where \( \alpha, \beta \in R \setminus \{0\} \). Then \( f \in C^{(1)} \) and \( f \in \overline{(IC)} \). If, moreover, we assume that \( \alpha \) and \( \beta \) are incommensurable, then the function \( f \) is not periodic.

**1.4. STEKLOV FUNCTIONS**

We first recall the basic notation related to Steklov functions.

For a given positive number \( h \) and for a function \( f : R \to R \) which is locally integrable, put

\[ S_f(h)(u) = \frac{1}{2h} \int_{u-h}^{u+h} f(s)ds \quad \text{for} \quad u \in R. \]

Then an \( S_f(h) \) is called the Steklov function of an \( f \).

It is easy to see that it follows:

**Theorem 1.23.** The following statements hold:

(i) If \( f \) is an \( (IC) \)-a.p. function, then the Steklov function \( S_f(h) \) is an \( (IC) \)-a.p. function and a \( C^{(1)} \) - a.p. function (see [1]).

(ii) If \( f \) is an \( (IC) \)-continuous function, then \( \lim_{h \to 0} (ID)(f, S_f(h)) = 0 \).

**Proof.** (i) We assume that \( f \in \overline{(IC)} \). Then, by Remark 1.5, \( f \) and \( F \) are \( B \)-a.p. functions. Moreover, we know that \( S_f(h) \) is a uniformly a.p. function,
as well (see [5]). Let us still remark that the indefinite integral \( F_{S_f(h)} \) of an
\( S_f(h) \) is bounded, from where, according to Remark 1.6, we obtain
\( S_f(h) \in (IC) \). The function \( S_f(h) \) is \( C^{(1)} \) - a.p. (see [2]), too.

(ii) In the same way as in [5], for an arbitrary but fixed \( t \in R \) we write

\[
| f(t) - S_f(h)(t) | \leq \frac{1}{2h} \int_{-h}^{h} | f(t) - f(s + t) | ds,
\]

\[
| F(t) - F_{S_f(h)}(t) | \leq \frac{1}{2h} \int_{-h}^{h} \int_{0}^{t} (f(s) - f(s + x)) ds dx
\]

for \( h > 0 \). Since \( f \) is an \( (IC) \)-continuous function, so for each \( \varepsilon > 0 \) there exists a
\( \delta > 0 \) such that \( (ID)(f, f_s) \leq \varepsilon/3 \) for \( s \in R, \ |s| < \delta \). Thus for all \( t \in R \) and
\( h \in (0, \delta) \) we obtain \( (ID)(f, S_f(h)) \leq \varepsilon \), and the proof is complete.

**COROLLARY 1.24.** If \( f \) is an \( (IC) \)-a.p. function, then

\[
\lim_{h \to 0} (ID)(f, S_f(h)) = 0.
\]

2. \((IC)^{(n)}\) - ALMOST PERIODIC FUNCTIONS

2.1. DEFINITIONS

We first present basic notations related to \((IC)^{(n)}\) -almost periodic functions.

Let \( N_0 \) denote the set \( N \cup \{0\} \).

By \( C^{(n)}(R) \) we denote the set of functions from \( R \) into itself with \( n \)-th
continuous derivatives on \( R \), for \( n \in N_0 \). For functions \( f, g \in C^{(n)}(R) \) we
define \((ID)^{(n)}\) – distans in the following

\[
(ID)^{(n)}(f, g) = \sup_{t \in R} \left( | f(t) - g(t) | + \sum_{i=1}^{n} | f^{(i)}(t) - g^{(i)}(t) | + T(f, g)(t) \right),
\]

where \( T \) is defined in the section 1.1.

We say that an \( f \in C^{(n)}(R) \) is \((IC)^{(n)}\)-bounded iff \( (ID)^{(n)}(f) < \infty \), where
\( (ID)^{(n)}(f) = (ID)^{(n)}(f, 0) \). We say that \( f \in C^{(n)}(R) \) is an \((IC)^{(n)}\)-continuous
function iff \( \lim_{h \to 0} (ID)^{(n)}(f, f_h) = 0 \). A sequence \((f_k)\) in \(C^{(n)}(R)\) will be called \((ID)^{(n)}\) -convergent to an \( f \in C^{(n)}(R) \) iff \( \lim_{k \to 0} (ID)^{(n)}(f, f_k) = 0 \).

Similarly as in Theorem 1.1, we obtain:

**Theorem 2.1.** If a sequence \((f_k)\) of \((IC)^{(n)}\)-continuous functions is \((ID)^{(n)}\) -convergent to a function \( f \in C^{(n)}(R) \), then an \( f \) is \((IC)^{(n)}\)-continuous.

A function \( f \in C^{(n)}(R) \) is called \((IC)^{(n)}\) – almost periodic ((\(IC)^{(n)}\) – a.p.) iff an \( f \) is \((IC)\)-a.p. and \( C^{(n)} \) – a.p. (see [1]), \( n \in N_0 \). By \((IC)^{(n)}\) we denote the set of \((IC)^{(n)}\) – a.p. functions, i.e. we have

\[
(\overline{(IC)^{(n)}}) = (\overline{IC}) \cap \overline{C^{(n)}}.
\]

For an arbitrary \( n \in N_0 \) every \((IC)^{(n+1)}\) – a.p. function is \((IC)^{(n)}\) – a.p. Moreover, every \((IC)^{(1)}\) – a.p. function is an \(L\)-a.p. function (see [1], [8]).

**2.2. Basic Properties**

Properties of \((IC)^{(n)}\) – a.p. functions we obtain using known theorems related to \((IC)\)-a.p. and \(C^{(n)} \) – a.p. functions (see [1], [2]).

**Theorem 2.2.** If \( f \) is an \((IC)^{(n)}\) – a.p. function, then:

(i) an \( f \) is \((IC)^{(n)}\) – bounded,

(ii) an \( f \) is \((IC)^{(n)}\) – continuous.

**Proposition 2.3.** If a sequence \((f_k)\) of \((IC)^{(n)}\) – a.p. functions \((ID)^{(n)}\) – convergent to a function \( f \in C^{(n)}(R) \), then an \( f \) is \((IC)^{(n)}\) – a.p.

**Theorem 2.4.** The following statements hold:

(i) A linear combination of \((IC)^{(n)}\) – a.p. functions is an \((IC)^{(n)}\) – a.p. function.

(ii) A product of two \(C^{(n)} \) – a.p. functions is \((IC)^{(n)}\) – a.p. if and only if the indefinite integral of a product of these functions is bounded.
(iii) If the derivative \( f^{(n+1)} \) of a \( C^{(n)} \)-a.p. function \( f \) is uniformly continuous, then the derivative \( f' \) is an \( (IC)^{(n)} \)-a.p. function.

(iv) If \( f \) is an \( (IC)^{(n)} \)-a.p. function and the indefinite integral \( F \) of \( f \), where \( F \) is the indefinite integral of \( f \), is bounded, then \( F \) is \( (IC)^{(n+1)} \)-a.p. function.

Now, we shall be occupied with \( (IC)^{(n)} \)-a.periodicity of a function \( f \).

**Theorem 2.5.** The following statements hold:

(i) If the derivative \( f^{(n)} \) of a function \( f \) is uniformly continuous and the indefinite integral \( F \) of \( f \) is \( C^{(n)} \)-a.p., then \( f \) is an \( (IC)^{(n)} \)-a.p. function.

(ii) If the derivative \( f' \) is an \( (IC)^{(n)} \)-a.p. function and the indefinite integral \( F \) of a function \( f \) is bounded, then \( f \) is an \( (IC)^{(n+1)} \)-a.p. function.

Finally, we shall give an example of an \( (IC)^{(n)} \)-a.p. function.

**Example 2.6.** Let \( f \) be the function defined by \( f(x) = \cos(\alpha x) + \sin(\beta x) \) for \( x \in \mathbb{R} \), where \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \) are incommensurable. Then \( f \in C^{(n)} \) and \( f \in (IC) \), because \( F \) is a bounded function. Thus \( f \in C^{(n)}(\mathbb{R}) \) is a \( (IC)^{(n)} \)-a.p. function, but not periodic.

### 2.3. Steklov Functions

In this section we shall give the theorem on \( (IC)^{(n)} \)-periodicity of Steklov functions. Using Theorem 1.23 and the theorem on approximation of \( C^{(n)} \)-a.p. functions by their Steklov functions (see [2]), similarly as in the part 1.4, we obtain:

**Theorem 2.7.** The following statements hold:

(i) If \( f \) is an \( (IC)^{(n)} \)-a.p. function, then the Steklov function \( S_f(h) \) is an \( (IC)^{(n+1)} \)-a.p. function.

(ii) If \( f \) is an \( (IC)^{(n)} \)-continuous function, then \( \lim_{h \to 0} (ID)^{(n)}(f, S_f(h)) = 0 \).
COROLLARY 2.8. If $f$ is an $(IC)^{(n)}_{\sigma}$ - a.p. function, then

$$\lim_{h \to 0} (ID)^{(a)}_{\sigma}(f, S_f(h)) = 0.$$ 

3. $(IC)^{(a)}_{\sigma}$ – ALMOST PERIODIC FUNCTIONS

3.1. DEFINITIONS

We first recall basic notations related to $(IC)^{(a)}_{\sigma}$ – almost periodic functions.

By $C^{(a)}(R)$ we denote the set of functions from $R$ into itself which have every derivatives. For functions $f, g \in C^{(a)}(R)$ and a sequence $a = (a_i)$ such that $a_i > 0$, $i = 1, 2, ..., $ we define $(ID)^{(a)}_{\sigma}$ – distans in the following

$$(ID)^{(a)}_{\sigma}(f, g) = \sup_{t \in R} \left( |f(t) - g(t)| + \sum_{i=1}^{\infty} a_i |f^{(i)}(t) - g^{(i)}(t)| + T(f, g)(t) \right),$$

where $T$ is defined in the section 1.1.

We say that an $f \in C^{(a)}(R)$ is $(IC)^{(a)}_{\sigma}$ – bounded iif $(ID)^{(a)}_{\sigma}(f) < \infty$, where $(ID)^{(a)}_{\sigma}(f) = (ID)^{(a)}_{\sigma}(f, 0)$. We say that $f \in C^{(a)}(R)$ is an $(IC)^{(a)}_{\sigma}$ – continuous function iif $\lim_{h \to 0} (ID)^{(a)}_{\sigma}(f, f_h) = 0$. A sequence $(f_k)$ in $C^{(a)}(R)$ will be called $(ID)^{(a)}_{\sigma}$ – convergent to an $f \in C^{(a)}(R)$, iif $\lim_{h \to 0} (ID)^{(a)}_{\sigma}(f, f_k) = 0$.

Similarly as in Theorem 1.1, we get:

THEOREM 3.1. If a sequence $(f_k)$ of $(IC)^{(a)}_{\sigma}$ – continuous functions is $(ID)^{(a)}_{\sigma}$ – convergent to a function $f \in C^{(a)}(R)$, then an $f$ is $(IC)^{(a)}_{\sigma}$ – continuous.

We say that an $f \in C^{(a)}(R)$ is conditionally locally bounded with respect to a sequence $a = (a_i)$: $a_i > 0$, $a_{i+1} \leq a_i$, $i = 1, 2, ...$, (i.e. $f \in (CBC^{(a)}_{a, loc})$) iif for an arbitrary closed interval $<x, y>$ and for each $i = 0, 1, 2, ...$ there exists a positive number $M_i = M_{i, f}^{<x, y>}$ such that
\[
\max_{t \in \mathbb{R}} |f^{(i)}(t)| = M_i \quad \text{and} \quad \sum_{i=1}^{\infty} a_i M_{i+1} < \infty.
\]

A function \( f \in (C^b)_{a,loc}^{(\infty)} \) is called \((IC)_{a}^{(\infty)}\) - almost periodic \(((IC)_{a}^{(\infty)}\) - a.p.) iff an \( f \) is \((IC)\) - a.p. and \((C)_{a}^{(\infty)}\) - a.p. (see [3]). By \((IC)_{a}^{(\infty)}\) we denote the set of \((IC)_{a}^{(\infty)}\) - a.p. functions, i.e. we have 

\[
(IC)_{a}^{(\infty)} = (IC) \cap C_{a}^{(\infty)}.
\]

It is seen that every \((IC)_{a}^{(\infty)}\) - a.p. function is \((IC)_{a}^{(n)}\) - a.p., for an arbitrary fixed \( n \in \mathbb{N}_0 \). Every \((IC)_{a}^{(\infty)}\) - a.p. function is an \( L \)-a.p. function (see [3], [8]), as well.

3.2. BASIC PROPERTIES

Properties of \((IC)_{a}^{(\infty)}\) - a.p. functions we obtain in the same way as properties of \((IC)_{a}^{(n)}\) - a.p. functions, using known theorems related to \((IC)\) - a.p. and \( C_{a}^{(\infty)}\) - a.p. functions (see [3]).

**THEOREM 3.2.** The following statements hold:

(i) Every \((IC)_{a}^{(\infty)}\) - a.p. function is \((IC)_{a}^{(\infty)}\) - bounded and \((IC)_{a}^{(\infty)}\) - continuous.

(ii) A linear combination of \((IC)_{a}^{(\infty)}\) - a.p. functions is an \((IC)_{a}^{(\infty)}\) - a.p. function.

(iii) Let be given the sequence \( b = (b_i): b_i = a_{i+1}^2 / c_i \) with \( c_i > 0, \ c_i \leq c_{i+1} \), \( c_i \geq \left( \frac{i+1}{((i+1)/2)} \right) \), \( i = 1, 2, \ldots \), and \( \sum_{i=1}^{\infty} 2^i / c_i < \infty \). Moreover, let the indefinite integral of a product of two \((IC)_{a}^{(\infty)}\) - a.p. functions \( f, g \) is bounded. Then a product \( fg \) is an \((IC)_{b}^{(\infty)}\) - a.p. function.

(iv) If a sequence \( (f_k) \) of \((IC)_{a}^{(\infty)}\) - a.p. functions is \((ID)_{a}^{(\infty)}\) - convergent to a function \( f \in (C^b)_{a,loc}^{(\infty)} \), then an \( f \) is \((IC)_{a}^{(\infty)}\) - a.p.

(v) Let \( \sup \{a_i / a_{i+1} : i = 1, 2, \ldots \} < \infty \) and let \( f \) be a \( C_{a}^{(\infty)}\) - a.p. function. Then the derivative \( f' \) is an \((IC)_{a}^{(\infty)}\) - a.p. function.
(vi) If \( f \) is an \((IC)_a^{(\sigma)}\) – a.p. function and the indefinite integral \( F_F \) of an \( F \), where \( F \) is the indefinite integral of an \( f \), is bounded, then \( F \) is also an \((IC)_a^{(\sigma)}\) – a.p. function.

(vii) If the indefinite integral \( F \) of a function \( f \) is a \( C_a^{(\sigma)} \) – a.p. function and \( \sup \{ a_i/a_{i+1} : i = 1, 2, \ldots \} < \infty \) then \( f \) is \((IC)_a^{(\sigma)}\) – a.p.

(viii) If the derivative \( f' \) is an \((IC)_a^{(\sigma)}\) – a.p. function and the indefinite integral \( F \) of a function \( f \) is bounded, then \( f \) is an \((IC)_a^{(\sigma)}\) – a.p. function.

**Proof.** For example, we shall show (vi). Let \( f \in (IC)_a^{(\sigma)} \). Since an \( F \) is uniformly a.p. and the indefinite integral \( F_F \) is bounded, so, according to Remark 1.6, we get \( F \in (IC) \). Moreover, \( F \in C_a^{(\sigma)} \) as the bounded indefinite integral \( F \) of a \( C_a^{(\sigma)} \) – a.p. function \( f \) (see [3]). Consequently, an \( F \) is \((IC)_a^{(\sigma)}\) – a.p.

Now, we shall give an example of an \((IC)_a^{(\sigma)}\) – a.p. function.

**Example 3.3.** Let \( f \) be the function defined by \( f(x) = \sin(\alpha x) + \sin(\beta x) \) for \( x \in \mathbb{R} \), with \( \alpha, \beta \in (-1,1) \) which are incommensurable. Then we have \( f \in (CBC_a^{(\sigma)}) \) and \( f \in (IC)_a^{(\sigma)} \), for a positive sequence \( a = (a_i) \) such that \( \sum_{i=1}^{\infty} a_i < \infty \). Moreover, \( f \in (IC) \), since \( F \) is bounded. Thus we obtain that \( f \) is an \((IC)_a^{(\sigma)}\) – a.p. function, but not periodic.

### 3.3. Steklov Functions

Finally, we shall be occupied with \((IC)_a^{(\sigma)}\) – a.periodicity of Steklov functions. Using Theorem 1.23 and the theorem on approximation of \( C_a^{(\sigma)} \) – a.p. functions by their Steklov functions (see [3]), we obtain:

**Theorem 3.4.** The following statements hold:

(i) If \( f \) is an \((IC)_a^{(\sigma)}\) – a.p. function, then the Steklov function \( S_f(h) \) is also \((IC)_a^{(\sigma)}\) – a.p.

(ii) If \( f \) is an \((IC)_a^{(\sigma)}\) – continuous function, then \( \lim_{h \to 0} (ID)_a^{(\sigma)}(f, S_f(h)) = 0. \)
COROLLARY 3.5. If $f$ is an $(IC)^{(\omega)}_a - a.p.$ function, then
\[
\lim_{h \to 0} (ID)^{(\omega)}_a (f, S_f(h)) = 0.
\]

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