ON A RADIAL PERIODIC SOLUTION TO THE DIFFUSION EQUATION FOR THE EXTERIOR OF A BALL

ABSTRACT: The subject of the paper is the construction of a periodic radial solution to the diffusion parabolic differential equation (1) \( \Delta u(x,t) - D_t u(x,t) = f(x,t) \), where \( \delta = D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2 \), \( x = (x_1, x_2, x_3) \), \( t \in (-\infty, \infty) \), in radial coordinates \((r,t)\), \( r = |x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \), in the exterior of the ball \( D_r \), \( \{ (r,t) : r > R, t \in (-\infty, \infty) \} \).

Equation (I) is of the from (II) \( D_r^2(W(r,t)) - D_t(W(r,t)) = 0 \), \( W(r,t) = rU(r,t) \). By the suitable Green function \( (r,t,p,s) \rightarrow G(r,t,p,s) \), we construct the periodic solution with respect to the variable \( t \) of equation (II) as the potential \( W(r,t) \) of the double layer.

KEY WORDS: a boundary-value problem, a periodic solution, Green function, Green potentials.

1. INTRODUCTION

The subject of the paper is the construction of a periodic solution to the equation

\[
PW(r,t) = D_r^2W(r,t) - D_tW(r,t) = 0
\]

in the domain

\( D = \{ (r,t) : r \in (R,\infty), t \in (-\infty,\infty) \} \).

To the construction of the solution we apply the suitable Green function \( G \). We suppose that solution is the potential of the double layer:

\[
W_1(r,t) = \int_{-\infty}^{t} H(s)(t-s)^{-3/2}(r-p)\exp\left(\frac{-(r-p)^2}{4(t-s)}\right)ds.
\]

In [1], the similar problem for the equation \( (D_r^2 - D_t)u(r,t) = 0 \) and for the strip is treated. In [3], periodic solution of a parabolic problem is studied.

2. GREEN FUNCTION \( G \)

Let

\[
U(r,t,p,s) = A(t-s)^{-1/2}\exp(B(t,s)(r-p)^2),
\]
where
\[ A = (2\sqrt{\pi})^{-1} \quad \text{and} \quad B(t, s) = (-4(t - s))^{-1}, \]
denote the fundamental solution to the equation \( PU = D_r^2 U - D_t U = 0. \)

By [2], the function
\[ (r, t, p, s) \rightarrow G(r, t, p, s) = U(r, t, p, s) - U(2R - r, t, p, s), \quad (r, t, p, s) \in D_1, \]
where \( D_1 = \{(r, t, p, s) : -\infty < s < t, \ t \in (-\infty, \infty), \ r > R, \ p > R, \ r \neq p\} \), is the Green function to the equation
\[ PG(r, t, p, s) = 0, \]
to the half space \( r > R \) and to the Dirichlet boundary-value conditions:
\[ G(R, t, p, s) = G(\infty, t, p, s) = 0. \]

3. Green potentials

Let us consider the Green potential
\[ W(r, t) = \int_{-\infty}^{t} H(s) D_p G(r, t, R, s) ds \]
and the potential of the double layer
\[ W_1(r, t) = \int_{-\infty}^{t} H(s) (t - s)^{-3/2} (r - p) \exp \left( -\frac{(r - p)^2}{4(t - s)} \right) ds. \]

4. Motivation of the problem

Consider the solution \( (r, t, T) \rightarrow W(r, t, T) \) of the Cauchy problem to the equation \( PW(r, t) = 0. \) By [1], the solution is of the form
\[ W(r, t, T) = \int_{R}^{\infty} W(p, T) G(r, t, p, T) dp, \]
where
\[ PU = D_r^2 U - D_t U = 0. \]

Let \( M(T) = \sup_{p \in (R, \infty)} |W(p, T)| \) and let \((K)\) denote the class of all functions \( W \) for which \( M(T) \) is bounded.
**Lemma.** If \( W \in (K) \) then:

1° the inequality

\[
|W(r,t,T)| \leq M(T) \int_R^T (t-T)^{-1/2} \exp \left( -\frac{(r-p)^2}{4(t-T)} \right) \, dp
\]

holds,

2° \( W(r,t) \to 0 \) as \( T \to -\infty \) for every \( t \in (-\infty, \infty) \).

**Proof.** 1°: Since \( G \geq 0 \) thus \( G \leq U(r,t,r,T) \) and we obtain 1°.

2° is a consequence of 1°.

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**5. Properties of the Potential \( W \)**

Let us consider the Green potential

\[
W_i(r,t) = \int_{-\infty}^t H(s)(t-s)^{-3/2} (r-R) \exp \left( -\frac{(r-R)^2}{4(t-s)} \right) \, ds.
\]

Denote by \((K_w)\) the class of all functions \( t \to H(t) \) continuous and bounded for \( t \in (-\infty, \infty) \) with the period \( w \).

**Theorem.** If \( H \in (K_w) \) then:

1° \( PW_i(r,t) = 0, \ (r,t) \in D \),

2° \( W_i(r,t) \to H(t) \) as \( (r,t) \to (R,t) \),

3° \( W_i(r,t) \to 0 \) as \( (r,t) \to (\infty,t) \) uniformly for every \( t \in (-\infty, \infty) \),

4° \((r,t) \to W_i(r,t)\) is the periodic function with respect to \( t \) with the period \( w \).

**Proof.** 1°: BY [2], we obtain 1°, 2°.

3°: We have

\[
W_i(r,t) = \int_{-\infty}^t H(s) \frac{(r-R)(r-R)^3}{(t-s)^{3/2} (r-R)^3} \exp(B(t,s)(r-R)^2) \, ds.
\]

By the last formula, we get the inequality

\[
|W_i(r,t)| \leq c(r-R)^{-2} \int_{-\infty}^t H(s) \frac{(r-R)^3}{(t-s)^{3/2}} \exp(B(t,s)(r-R)^2) \, ds.
\]
Supplying in the last integral the change of the integral variable

\[ z = \frac{r - R}{(t - s)^{1/2}}, \quad dz = -\frac{2(r - R)}{(t - s)^{3/2}} ds, \]

we obtain

\[ |W_1(r, t)| \leq c \int_0^\infty z^3 \exp(-z^2) dz \to 0 \quad \text{as} \quad r \to \infty \]

with \( c = \sup_{s \in (-\infty, \infty)} |H(s)|. \)

4°: We have

\[ |W_1(r, t + w)| = \int_{-\infty}^{t+w} H(s)D_\rho G(r - R, t + w - s) ds. \]

Applying in the last integral the change of the integral variables

\[ s = w + z, \quad ds = dz, \quad z \in (-\infty, \infty), \]

we obtain

\[ W_1(r, t + w) = \int_{-\infty}^{t} H(w + z)D_\rho G(r - R, t - z) dz = W_1(r, w). \]

REFERENCES


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Received on 02.07.2001 and, in revised form, on 08.10.2001.