ALEKSANDER WASZAK

ON THE STRONG CONVERGENCE IN SOME SEQUENCE SPACES

ABSTRACT: The purpose of this paper is to introduce and study an idea of lacunary strong \((A, \varphi)\)-convergence with respect to a modulus function. In courses of these investigations we study some connections between \((A, \varphi)\)-strong summability of sequences and lacunary strong convergence with respect to a modulus or lacunary statistical convergence.

KEY WORDS: sequence spaces, modular spaces.

1. INTRODUCTION

In papers of J. Musielak [9], J. Musielak and W. Orlicz [12], W. Orlicz [15] and myself [18] there are considered and investigated some modular spaces connected with strong \((A, \varphi)\)-summability of sequences.

The spaces \(N_\Theta\) of lacunary strong convergence of sequences have been introduced by A. Freedman, J. Somberg and M. Raphel [4], where

\[
N_\Theta = \left\{ x = (t_\nu) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{\nu \in I_r} |t_\nu - s| = 0 \text{ for some } s \right\}
\]

and \(\Theta = (k_\nu)\) is a given lacunary sequence.

If \(f\) is a given modulus function (which have been introduced by H. Nakano [14]) and \(A = (a_{n\nu})\) is a given matrix, then applying the concept of T. Bilging [1] we may define the sequence space (compare e.g. [2], [3] or [8])

\[
N_\Theta(A, f) = \left\{ x = (t_\nu) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{\nu \in I_r} f\left( \sum_{\nu = 1}^\infty a_{n\nu} t_\nu - s \right) = 0 \text{ for some } s \right\}
\]

Throughout this paper it will be supposed that \(s = 0\) and that we take the sequence \((\sigma_\nu \varphi)\), where \(\sigma_\nu \varphi(x) = \sum_{\nu = 1}^\infty a_{n\nu} \varphi(|t_\nu|)\) instead of the sequence \(\sum_{\nu = 1}^\infty a_{n\nu} t_\nu\).

Finally, the space \(T_\Theta((A, \varphi), f)\) of lacunary strongly convergent sequences is defined by the formula
\[ T_\Theta((A, \varphi), f) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} f(|\sigma_n^\Theta(x)|) = 0 \right\}. \]

2. PRELIMINARIES

Let \( T, T_b, T_0, T_f \) denote spaces of all real sequences, bounded real sequences, real sequences convergent to zero and sequences with a finite number of elements different from zero, respectively. Sequences belonging to \( T \) will be denoted by \( x = (t_v), \ y = (s_v), \ x_m = (t_v^m), \ |x| = (|t_v|), \ 0 = (0) \) and \( x^q \) will mean the sequence \( t_1, t_2, \ldots, t_q, 0, 0, \ldots \). Moreover, we shall write \( e_p, e^q, e^p_q \) for the following sequences: \( 0, 0, \ldots, 1, 0, \ldots \) (with 1 at the \( p \) th place); \( 1, 1, \ldots, 1, 0, \ldots \) (with 1 at the first \( q \) places); \( 0, 0, 1, 0, 0, 1, 0, \ldots \) (with 1 at the \( p \) th, \( (p + 1) \) st, \( (p + q - 1) \) st place), respectively.

A sequence of positive integers \( \Theta = (k_r) \) is called lacunary if \( k_0 = 0, k_r < k_{r+1} \) for all \( r \) and if \( I_r = [k_{r-1}, k_r) \) then \( h_r = k_r - k_{r-1} \to 0 \) as \( r \to \infty \). In the following the quotient \( k_r/k_{r-1} \) will be denoted by \( q_r \), (compare [4]).

Let \( A = (a_{nv}) \) be an infinite matrix such that:

a) is nonnegative i.e. \( a_{nv} \geq 0 \) for \( n, v = 1, 2, \ldots, \)

b) for an arbitrary positive integer \( n \) (or \( v \)) there exists a positive integer \( v_0 \)

(or \( n_0 \)) such that \( a_{nv_0} \neq 0 \) (or \( a_{nv} \neq 0 \)), respectively,

c) there exist \( \lim_{n \to \infty} a_{nv} = 0 \) for \( v = 1, 2, \ldots, \)

d) \( \sup_n \sum_{v=1}^{\infty} a_{nv} = K < \infty \),

e) \( \sup_n a_{nv} \to 0 \) as \( v \to \infty \).

By a \( \varphi \)-function we understood a continuous non-decreasing function \( \varphi(u) \) defined for \( u \geq 0 \) and such that \( \varphi(0) = 0, \ \varphi(u) > 0 \) for \( u > 0 \) and \( \varphi(u) \to \infty \) as \( u \to \infty \). The symbol \( \varphi(|x|) \) means the function \( \varphi(|x(t)|) \).

A \( \varphi \)-function \( \varphi \) is called non weaker then a \( \varphi \)-function \( \psi \) and we write \( \psi \preceq \varphi \) if there are constants \( c, b, k, l > 0 \) such that \( c \psi(lu) \leq b \varphi(ku) \), (for all, large or small \( u \), respectively).

\( \varphi \)-functions \( \varphi \) and \( \psi \) are called equivalent and we write \( \varphi \sim \psi \) if there are positive constants \( b_1, b_2, c, k_1, k_2, l \) such that \( b_1 \varphi(k_1u) \leq c \psi(lu) \leq b_2 \varphi(k_2u) \), (for all, large or small \( u \), respectively).
A \varphi\text{-function} \varphi is said to satisfy the condition (\Delta_2), (for all, large or small \( u \), respectively) if for some constant \( k > 1 \) there is satisfied the inequality \( \varphi(2u) \leq k\varphi(u) \). For more properties of \varphi\text{-function} see e.g. [7], [10], [11].

By a modulus function we understood the increasing function \( f \) from \([0, \infty)\) to \([0, \infty)\) such that: \( f(x) = 0 \) if and only if \( x = 0 \), \( f(x + y) \leq f(x) + f(y) \) for \( x, y \geq 0 \) and is continuous from the right at 0, (compare [14]).

3. SPACES OF STRONGLY \((A, \varphi)\) – SUMMABLE SEQUENCES

For a given \varphi\text{-function} \( \varphi(u) \) and the matrix \( A = (a_{nv}) \) we adopt the following notation:

\[
\begin{align*}
\sigma_n^{\varphi}(x) &= \sum_{\nu=1}^{\infty} a_{nv} \varphi([t_{\nu}]) \quad \text{for} \ n = 1, 2, ..., \\
T_0 = \left\{ x \in T : \sigma_n^{\varphi}(x) < \infty \quad \text{for} \ n = 1, 2, ... \quad \text{and} \quad \lim_{n \to \infty} \sigma_n^{\varphi}(x) = 0 \right\}, \\
T^0 = \left\{ x \in T : \lambda x \in T_0 \quad \text{for an arbitrary} \ \lambda > 0 \right\}, \\
T^* = \left\{ x \in T : \lambda x \in T_0 \quad \text{for a certain} \ \lambda > 0 \right\}.
\end{align*}
\]

Sequences \( x \) belonging to \( T^* \) are called strongly \((A, \varphi)\) – summable to zero.

A list of the most interesting properties concerning the space \( T^* \) is presented below, (compare also [9], [12], [15] or [18]).

1. \( T_f \subset T^\varphi, \ T_\varphi \subset T^0 \subset T^* \).
2. If \( a_{nv} \to 0 \) as \( n \to \infty \) for all \( v \), then \( e^p, e^q, e^q_\varphi \in T_\varphi \).
3. For an arbitrary \varphi\text{-function} \varphi we have \( T_b \cap T^\varphi = T_b \cap T^* \).
4. For arbitrary two \varphi\text{-function} \varphi and \psi the following identity holds
   \( T_b \cap T^\varphi = T_b \cap T^\psi \).
5. If \( \psi < \varphi \) then \( T_\varphi \subset T_\psi \) and \( T^* \subset T^* \).
6. If the \varphi\text{-function} \( \varphi(u) \) satisfiess the condition \((\Delta_2)\) then \( T_\varphi = T^* \).

4. SPACES OF LACUNARY STRONGLY CONVERGENT SEQUENCES

Let \( \varphi \) and \( f \) be given \varphi\text{-function} and modulus function, respectively. Moreover, let a matrix \( A \) and a lacunary sequence \( \Theta \) be given. We introduce sequence space \( T_\Theta((A, \varphi), f) \) by the formula:
\[ T_\Theta((A, \varphi), f) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{nv} \varphi(|t_v|) \right) = 0 \right\}. \]

The sequence \( x \) is said to be lacunary strong \((A, \varphi)\)-convergent to zero with respect to a modulus \( f \), if \( x \in T_\Theta((A, \varphi), f) \).

Let us remark that in particular we have:

1° If \( \varphi(u) = u \) for all \( u \), then we obtain the space

\[ N^0_\Theta(A, f) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{nv} t_v \right) = 0 \right\}, \]

which was defined and considered in [1].

2° If \( f(v) = v \) then \( T_\Theta((A, \varphi), v) = T_\Theta((A, \varphi)) \), where

\[ T_\Theta((A, \varphi)) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} \sum_{v=1}^{\infty} a_{nv} \varphi(|t_v|) = 0 \right\}. \]

3° If \( A = I \) then we obtain the following sequence space

\[ T_\Theta((I, \varphi), f) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} f(\varphi(|t_v|)) = 0 \right\}. \]

4° If \( A = I \) and moreover \( \varphi(u) = u \) and \( f(v) = v \) for all \( u \) and \( v \), respectively, then we have the sequence space

\[ N^0_\Theta = T_\Theta((I, u), v) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} t_v = 0 \right\}, \]

(compare [1]).

5° If the matrix \( A = (a_{nv}) \) is defined by the formula:

\[ a_{nv} = \frac{1}{n} \text{ for } n \geq v \text{ and } a_{nv} = 0 \text{ for } n < v, \]

then applying the properties of \( \Theta \) and \( f \) we obtain the sequence space

\[ T_\Theta((A, \varphi), f) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} f \left( \frac{1}{n} \sum_{v=1}^{n} \varphi(|t_v|) \right) = 0 \right\}. \]

Moreover, we have the following inequalities
\[
\frac{1}{h_r} f \left( \frac{1}{q_r} \min_{1 \leq v \leq k_{r-1}} \{ \varphi(\| t_v \|) \} \right) \leq \frac{1}{h_r} \sum_{n \in \mathbb{N}} f \left( \frac{1}{n} \sum_{v=1}^n \varphi(\| t_v \|) \right) \leq f \left( \frac{1}{k_{r-1} + 1} \sum_{v=1}^{k_r} \varphi(\| t_v \|) \right).
\]

**Theorem 1.** Let \( f \) be a any modulus function and let \( \varphi \) function \( \varphi \), the matrix \( A \) and the sequence \( \Theta \) be given. If

\[
w((A, \varphi), f) = \left\{ x = (t_v) : \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^m f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right) = 0 \right\}
\]

then the following relations are true:

(a) If \( \liminf_{r} q_r > 1 \), then we have \( w((A, \varphi), f) \subseteq T_\Theta ((A, \varphi), f) \).

(b) If \( \limsup_{r} q_r < \infty \), then we have \( T_\Theta ((A, \varphi), f) \subseteq w((A, \varphi), f) \).

(c) If \( 1 < \liminf_{r} q_r \leq \limsup_{r} q_r < \infty \), then \( T_\Theta ((A, \varphi), f) = w((A, \varphi), f) \).

**Proof.** (a). Let us supoese that \( x \in w((A, \varphi), f) \). There exists \( \delta > 0 \) such that \( q_r > 1 + \delta \) for sufficiently large \( r \) and we have \( h_r/k_r \geq \delta/(1 + \delta) \) for sufficiently large \( r \). Consequently,

\[
\frac{1}{h_r} \sum_{n=1}^{k_r} f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right) \geq \frac{1}{k_r} \sum_{n \in \mathbb{N}} f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right) = \frac{h_r}{k_r} \frac{1}{h_r} \sum_{n \in \mathbb{N}} f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right) \geq \frac{\delta}{1 + \delta} \frac{1}{h_r} \sum_{n \in \mathbb{N}} f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right).
\]

Finally, \( x \in T_\Theta ((A, \varphi), f) \).

**Proof**. (b). The condition \( \limsup_{r} q_r < \infty \) implies that there exists a constant \( M > 0 \) such that \( q_r < M \) for every \( r \). If \( x \in T_\Theta ((A, \varphi), f) \) and \( \varepsilon \) is an arbitrar positive number, then there exists an index \( m_0 \) such that

\[
H_{m_0} = \frac{1}{h_m} \sum_{n \in \mathbb{N}} f \left( \left| \sum_{v=1}^n a_{nv} \varphi(\| t_v \|) \right| \right) < \varepsilon.
\]
for every $m \geq m_0$. Thus, we can find a positive constant $L$ such that $H_m \leq L$ for all $m$. In the following choosing an integer $\alpha$ such that $k_{r-1} < \alpha < k_r$ we obtain

$$I = \frac{1}{\alpha} \sum_{n=1}^{\alpha} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) \leq \frac{1}{k_{r-1}} \sum_{n=1}^{k_r} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{r-1}} \sum_{m=1}^{m_0} \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right),$$

$$I_2 = \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right).$$

It is easily verified that

$$I_1 = \frac{1}{k_{r-1}} \sum_{m=1}^{m_0} \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) =$$

$$= \frac{1}{k_{r-1}} \left( \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) \right) + \ldots + \sum_{n \in I_{m_0}} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) \leq$$

$$\leq \frac{1}{k_{r-1}} \left( h_1 H_1 + \ldots + h_{m_0} H_{m_0} \right) \leq \frac{1}{k_{r-1}} m_0 k_{m_0} \sup_{1 \leq i \leq m_0} H_i \leq \frac{m_0 k_{m_0}}{k_{r-1}} L.$$

Moreover, we have

$$I_2 = \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) =$$

$$= \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} \left( \frac{1}{h_m} \sum_{n \in I_m} f\left( \sum_{v=1}^{\infty} a_{n,v} \varphi(\{t_v\}) \right) \right) h_m \leq$$

$$\leq \varepsilon \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{\alpha} h_m \leq \varepsilon \frac{k_r}{k_{r-1}} = \varepsilon q_r < \varepsilon M.$$
5. PROPERTIES AND THEOREMS

Let the sequence $\Theta$, the modulus function $f$ be given and let $\phi$ and $\psi$ are two $\phi$-functions.

**THEOREM 2.** Let us suppose that the matrix $A$ satisfies the condition

$$a_{n1} + a_{n2} + ... \leq K \text{ for } n=1,2,...$$

and let $\phi$-functions $\phi$ and $\psi$ satisfy the condition $(\Delta_2)$ for large $u$.

(a) If $\psi < \phi$ then $T_\Theta((A, \phi), f) \subseteq T_\Theta((A, \psi), f)$.

(b) If $\phi$-function $\phi$ and $\psi$ are equivalent for large $u$, then $T_\Theta((A, \phi), f) = T_\Theta((A, \psi), f)$.

**PROOF.** Let $x=(t_v) \in T_\Theta((A, \phi), f)$. By assumption we have

$$\psi(|t_v|) \leq b \phi(c |t_v|)$$

for $b$, $c$, $u_0 > 0$ and $|t_v| > u_0$. Let us denote $x=x^1 + x^2$, where $x^1=(t^1_v)$ and $t^1_v = t_v$ for $|t_v| < u_0$ and $t^1_v = 0$ for remaining values of $v$. It is easily seen that $x^1 \in T_\Theta((A, \phi), f)$. Moreover, by the assumptions and the inequality $(\cdot)$ we get

$$\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{nv} \psi(|t^2_v|) \right) \leq \frac{1}{h_r} \sum_{n \in I_r} f \left( b \sum_{v=1}^{\infty} a_{nv} \phi(c |t^2_v|) \right) \leq \frac{L}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{nv} \phi(|t^2_v|) \right),$$

where the constant $L$ is connected with properties of $f$ and $\phi$.

Finally, we obtain $x^2=(t^2_v) \in T_\Theta((A, \psi), f)$ and in consequence $x \in T_\Theta((A, \psi), f)$.

The identity $T_\Theta((A, \phi), f) = T_\Theta((A, \psi), f)$ is proved, analogously.

**THEOREM 3.** Let the $\phi$-function $\phi(u)$ satisfies the condition $(\Delta_2)$ and let the matrix $A$ has the property $a_{n1} + a_{n2} + ... \leq K$ for $n=1,2,...$. The following conditions are true:

(a) If $x=(t_v) \in T_\Theta((A, \phi), f)$ and $\alpha$ is an arbitrary number, then $\alpha x \in T_\Theta((A, \phi), f)$. 
(b) If \( x, y = (t_v) \in T_\theta ((A, \varphi), f) \) where \( x = (t_v), \ y = (s_v) \) and \( \alpha, \beta \) are given numbers, then \( \alpha x + \beta y \in T_\theta ((A, \varphi), f) \).

(c) \( T_\theta ((A, \varphi), f) \) is a linear space.

**Proof.** Let \( x \in T_\theta ((A, \varphi), f) \). First let us remark that for \( 0 < \alpha < 1 \) we get

\[
\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(\alpha |t_v|) \right) \leq \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(|t_v|) \right).
\]

Moreover, if \( \alpha > 1 \) then we may find a positive number \( s \) such that \( \alpha < 2^s \) and we obtain

\[
\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(\alpha |t_v|) \right) \leq \frac{1}{h_r} \sum_{n \in I_r} f \left( d^s \sum_{v=1}^{\infty} a_n \varphi(|t_v|) \right) \leq \frac{L}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(|t_v|) \right),
\]

where \( d \) and \( L \) are constants connected with the properties of \( \varphi \) and \( f \). Hence we obtain the condition (a).

In the following let the numbers \( \alpha, \beta \) and the elements \( x, y \in T_\theta ((A, \varphi), f) \) be given. From the part (a) it follows that the following inequality is true

\[
\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(|\alpha t_v + \beta s_v|) \right) \leq L_1 \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(|t_v|) \right) + L_2 \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_n \varphi(|s_v|) \right),
\]

where the constants \( L_1 \) and \( L_2 \) are defined as in (a). In consequence \( x + y \in T_\theta ((A, \varphi), f) \).

**Remark.** Let us remark that the modulus function \( f \) is continuous in the interval \([0, \infty)\). Moreover, it is easily verified that by the assumptions of matrix \( A \), the sums of elements in \( n \)-th row of the matrix \( A \)

\[
S^n_{pq} = a_{n,p} + a_{n,p+1} + \ldots + a_{n,p+q-1} \quad \text{and} \quad \sum_{n \in I_r} f(S^n_{pq})
\]

are bounded and tend to zero as \( n \to \infty \), (compare [13], [19]).
In consequence we have $e, e^q, e^q_p \in T_\Theta((A, \varphi), f)$.

**Theorem 5.** $T_\Theta((A, \varphi)) \subseteq T_\Theta((A, \varphi), f)$.

**Proof.** Let $x \in T_\Theta((A, \varphi))$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(\nu) < \varepsilon$ for every $\nu \in [0, \delta]$. We can write

$$\frac{1}{h_r} \sum_{\nu \in I_r} f\left(\sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|)\right) = S_1 + S_2,$$

where $S_1 = \frac{1}{h_r} \sum_{\nu \in I_r} f\left(\sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|)\right)$ and this sum is taken over $\sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \leq \delta$ and $S_2 = \frac{1}{h_r} \sum_{\nu \in I_r} f\left(\sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|)\right)$ and this sum is taken over $\sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|) > \delta$.

By definition of the modulus $f$ we have $S_1 = \frac{1}{h_r} \sum_{\nu \in I_r} f(\delta) = f(\delta) < \varepsilon$ and moreover $S_2 = f(1)\frac{1}{\delta} \frac{1}{h_r} \sum_{\nu \in I_r} \sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|)$. Finally, we get $x \in T_\Theta((A, \varphi), f)$.

6. **Some Remarks on Lacunary $(A, \varphi)$–Statistical Convergence**

Let $\Theta$ be a lacunary sequence, and let the matrix $A = (a_{n\nu})$, the sequence $x = (t_\nu)$, the $\varphi$-function $\varphi(u)$ and a positive number $\varepsilon$ be given. We adopt the following notation

$$K_\Theta^r((A, \varphi), \varepsilon) = \left\{ n \in I_r : \sum_{\nu = 1}^{\infty} a_{n\nu} \varphi(|t_\nu|) \geq \varepsilon \right\}.$$

The sequence $x$ is said to be lacunary $(A, \varphi)$–statistically convergent to a number zero if for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{k_r} \mu(K_\Theta^r((A, \varphi), \varepsilon)) = 0,$$
where $\mu(K^r_\Theta((A, \varphi), \varepsilon))$ denotes the number of elements belonging to $K^r_\Theta((A, \varphi), \varepsilon)$. The set of all lacunary $(A, \varphi)$–statistical convergent sequences is denoted by $S_\Theta((A, \varphi))$,

\[
S_\Theta((A, \varphi)) = \left\{ x = (t_v) : \lim_{r \to \infty} \frac{1}{h_r} \mu(K^r_\Theta((A, \varphi), \varepsilon)) = 0 \right\},
\]

(compare [2], [4], [5], [6], and [17]).

**THEOREM 6.** If $\psi < \varphi$ then $S_\Theta((A, \psi)) \subset S_\Theta((A, \varphi))$.

**PROOF.** By assumptions we have $\psi(|t_v|) \leq b \varphi(c|t_v|)$ and we have

\[
\sum_{v=1}^{\infty} a_{nv} \psi(|t_v|) \leq b \sum_{v=1}^{\infty} a_{nv} \varphi(c|t_v|) \leq L \sum_{v=1}^{\infty} a_{nv} \varphi(|t_v|),
\]

for $b, c > 0$, where the constant $L$ is connected with properties of $\varphi$. Thus, the condition $\sum_{v=1}^{\infty} a_{nv} \varphi(|t_v|) \geq \varepsilon$ implies the condition $\sum_{v=1}^{\infty} a_{nv} \varphi(|t_v|) \geq \varepsilon$ and in consequence we obtain

\[
\mu(K^r_\Theta((A, \varphi), \varepsilon)) \leq \mu(K^r_\Theta((A, \psi), \varepsilon))
\]

and

\[
\lim_{r \to \infty} \frac{1}{h_r} \mu\left(K^r_\Theta((A, \varphi), \varepsilon)\right) \leq \lim_{r \to \infty} \frac{1}{h_r} \mu\left(K^r_\Theta((A, \psi), \varepsilon)\right).
\]

**THEOREM 7.** If $\psi \sim \varphi$ then $S_\Theta((A, \varphi)) = S_\Theta((A, \psi))$.

**THEOREM 8.**

(a) If the matrix $A$, the sequence $\Theta$ and functions $f$ and $\varphi$ be given, then

\[
T_\Theta((A, \varphi), f) \subset S_\Theta((A, \varphi)).
\]

(b) If the $\varphi$–function $\varphi(u)$ and the matrix $A$ are given, and if the modulus function $f$ is bounded, then

\[
S_\Theta((A, \varphi)) \subset T_\Theta((A, \varphi), f).
\]

(c) If the $\varphi$–function $\varphi(u)$ and the matrix $A$ are given, and if the modulus function $f$ is bounded, then

\[
S_\Theta((A, \varphi)) = T_\Theta((A, \varphi), f).
\]
PROOF. (a) Let $f$ be a modulus function and let $\varepsilon$ be a positive number. We have the following inequalities

$$
\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \right) \geq \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \right) \geq \frac{1}{h_r} f(\varepsilon) \sum_{n \in I_r} 1 \geq
$$

$$
\geq \frac{1}{h_r} f(\varepsilon) \mu(K^r_{\Theta}((A, \varphi), \varepsilon)),
$$

where $I_r = \left\{ n \in I_r : \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \geq \varepsilon \right\}$. Finally, if $x \in T_0((A, \varphi), f)$ then $x \in S_\Theta((A, \varphi))$.

PROOF. (b) Let us suppose that $x \in S_\Theta((A, \varphi))$. If the modulus function $f$ is a bounded function, then there exists an integer $L$ such that $f(v) \leq L$ for all $v \geq 0$. In the following let

$$
I_r^2 = \left\{ n \in I_r : \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) < \varepsilon \right\}.
$$

Thus, we have

$$
\frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \right) \leq \frac{1}{h_r} \sum_{n \in I_r} f \left( \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \right) +
$$

$$
+ \frac{1}{h_r} \sum_{n \in I_r^2} f \left( \sum_{v=1}^{\infty} a_{n,v} \varphi(|t_v|) \right) \leq \frac{1}{h_r} L \mu(K^r_{\Theta}((A, \varphi), \varepsilon)) + f(\varepsilon).
$$

Taking the limit as $\varepsilon \to 0$, we obtain that $x \in T_0((A, \varphi), f)$.

PROOF of the part (c) follows from (a) and (b).

THEOREM 9. Let us suppose that the matrix $A$ is regular and that the modulus function $f$ is bounded. Then the condition $x \in T_0$ implies $x \in S_\Theta((A, \varphi))$.

PROOF. If $t_v \to 0$ as $v \to \infty$ then be regularity of $A$ and by the definition of statistical $(A, \varphi)$ - convergence we have
\[
\lim_{n \to \infty} \sum_{v=1}^{\infty} a_n \psi(|t_n|) = 0.
\]

Thus
\[
\lim_{r \to \infty} \frac{1}{h_r} \mu(K_{(A, \varphi), \varepsilon}) = 0.
\]

Finally, we obtain \( x \in T_{\Theta}((A, \varphi), f) \subset S_{\Theta}((A, \varphi)) \).

REFERENCES


( Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland)

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